

# Navier-Stokes equations under Marangoni boundary conditions generate all hyperbolic dynamics

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**ABSTRACT.** In this paper, we consider dynamics defined by the Navier-Stokes equations under the Marangoni boundary conditions in a two dimensional domain. This model of fluid dynamics involve fundamental physical effects: convection, diffusion and capillary forces. The main result is as follows: local semiflows, defined by the corresponding initial boundary value problem, can generate all possible structurally stable dynamics defined by  $C^1$  smooth vector fields on compact smooth manifolds (up to an orbital topological equivalence). To generate a prescribed dynamics, it is sufficient to adjust some parameters in the equations, namely, the Prandtl number and an external heat source.

## 1. Introduction

In this paper, we state an analytical proof of existence of strange attractors for a model of fluid dynamics. The hypothesis that turbulence can be connected with strange (chaotic) attractors was pioneered in papers [21, 22]. We consider the initial boundary values problem (IBVP) defined by the Navier-Stokes (NS) equations in the two-dimensional case with the Marangoni boundary conditions. These equations present a model for non-compressible fluid dynamics, which involves fundamental physical effects: convection, heat transfer and capillarity. This model describes surface driven the Bénard-Marangoni convection leading to interesting phenomena, for example, Bénard cells, (see [2, 9, 11] and references therein).

The main result of this paper can be outlined as follows: local semiflows, induced by the NS equations with the Marangoni boundary conditions, can generate all possible hyperbolic dynamics defined by  $C^1$ -smooth vector fields on finite dimensional compact smooth manifolds (up to an orbital topological equivalence). To generate a prescribed dynamics, it is sufficient to adjust some parameters in the NS equations, namely, the Prandtl number and a heat source in the heat transfer equation. The well known examples of hyperbolic dynamics with a "chaotic" behaviour are the Anosov flows, the Smale A-axiom systems and the Smale horseshoes [28, 15, 12].

Although numerous works were dedicated to the NS equations, ( for example, [33, 32, 19, 20]), nonetheless, up to now general results on dynamical complexity of dissipative dynamical systems defined by the NS equations do not exist. Results on chaos existence were obtained for quasilinear parabolic equations and reaction-diffusion systems [25, 26, 34] by a special approach. The main ingredient of this approach is the method of realization of vector fields (RVF) proposed by P. Poláčik

[25, 26]. Let us outline briefly the RVF method and some previously obtained results.

Let us consider an initial boundary value problem associated with a system of PDE's. Assume this problem involves some parameters  $\mathcal{P}$  (for example, in the Marangoni problem the parameters are a heat source and the Prandtl number). We obtain a family of local semiflows  $S_{\mathcal{P}}^t$  generated by these initial boundary value problems, where each semiflow depends on the corresponding parameter value  $\mathcal{P}$ . Suppose for an integer  $n > 0$ , there is an appropriate choice of the parameter  $\mathcal{P}_n$  such that the corresponding initial boundary value problem generates a global semiflow  $S^t(\mathcal{P}_n)$  possessing an  $n$ -dimensional finite  $C^1$ -smooth positively invariant manifold  $\mathcal{M}_n$  (we can suppose, for simplicity, that this manifold is diffeomorphic to the unit ball  $\mathcal{B}^n \subset \mathbf{R}^n$ ). The semiflow  $S^t(\mathcal{P}_n)$ , restricted to  $\mathcal{M}_n$  (a local inertial form), is defined by a  $C^1$ -vector field  $Q$  on  $\mathcal{B}^n$ . Then we say that the family  $S_{\mathcal{P}}^t$  realizes the vector field  $Q$ . We say that this family  $\epsilon$ -realizes a vector field  $\bar{Q}$  if the field  $Q$  is a  $\epsilon$ -perturbation of the field  $\bar{Q}$  in  $C^1(\mathcal{B}^n)$  (we consider  $C^1$ -norms in order to apply the theorem on persistence of hyperbolic sets [28, 15]).

By the RVF method, it has been shown that semiflows associated with some special quasilinear parabolic equations in two dimensional domains can generate complicated hyperbolic sets [26, 8]. For reaction-diffusion systems of a special form the RVF method allows to prove existence of chaotic regimes [34]. One can show that, for any integer  $n$ , semiflows induced by these systems can realize a dense set of the fields in the space of all  $C^1$ -smooth vector fields on  $\mathcal{B}^n$  [34]. Therefore, such systems generate any structurally stable (persistent under sufficiently small  $C^1$ -perturbations) dynamics, up to orbital topological equivalence [28, 15]. The corresponding families of the semiflows can be called *maximally dynamically complex*. If a family of semiflows enjoys this dynamical complexity property, this family generates all compact invariant hyperbolic dynamics on finite dimensional compact smooth manifolds.

By this terminology, the main result of this paper is as follows. The family of semiflows associated with our IBVP possesses the property of maximal dynamical complexity (see Theorem 3.2).

Using the RVF we encounter two main technical difficulties. First, in previous papers [25, 26, 8, 34] the RVF has been used only for systems with non-polynomial nonlinearities whereas the NS equations involve quadratic ones. In the first part of the paper we overcome this difficulty. We consider systems of differential equations with quadratic nonlinearities

$$(1.1) \quad \frac{dX}{dt} = K(X, X) + MX + f, \quad X \in \mathbf{R}^N,$$

where  $X(t)$  is a unknown function,  $X = (X_1, \dots, X_N) \in \mathbf{R}^N$ ,  $K(X, X)$  is a bilinear quadratic form,  $f \in \mathbf{R}^N$ ,  $M$  is a  $N \times N$  matrix. Systems (1.1) have important applications, in particular, in chemistry, where they describe bimolecular chemical reactions [39], and for population dynamics. Results on existence of complex dynamics for (1.1) were first obtained in [16] (see also [39]). In the paper [17] it is shown that, when we vary  $N, K, M$  and  $f$ , systems (1.1) realize all polynomial equations  $D^n z = p(z, Dz, \dots, D^{n-1}z)$  of  $n$ -th order for all  $n$ . The result [17] is based on purely algebraic methods whereas the work [16] uses slow manifolds. In [36] the RVF method is applied to investigate dynamics of systems (1.1). It is shown that

systems (1.1) generate a maximally complex family of semiflows, where parameters are  $N, M$ , the bilinear form  $K$  and  $f$ .

In this paper, we are dealing with a more complicated situation, when quadratic forms  $K(X, X)$  can not be considered as free parameters. This difficulty is not too hard and it can be overcome by the methods of the invariant manifold theory [13, 3, 4, 1, 32, 38, 37] that allows us to reduce systems (1.1) of large dimension to analogous systems of smaller dimension, where the corresponding  $K(X, X)$  can be considered as free parameters (see section 8 for more detail).

The second part of the proof resolves a much more sophisticated problem: how to reduce the dynamics defined by our IBVP to systems (1.1). Physically, this reduction describes dynamics of some "main modes"  $X$ . We reduce the Marangoni problem to system (1.1) by locally attracting invariant manifolds. In order to proceed it, we choose the parameters in the NS equations in a special way. It allows us to extract a finite set of "main modes". Note that it is impossible to prove existence of locally attracting invariant or inertial manifolds for the general NS equations, even for two dimensional case [4], therefore, we really need this special parameter choice. We proceed this extraction using spectral properties of a linear operator  $L$  that determines the linearization of the NS equations. This operator has  $N$  zero eigenvalues and all the rest spectrum of  $L$  lies in the negative half plane and it is separated by a positive barrier from the imaginary axis. To find the operator  $L$  with such spectral barrier (gap) is not easy, and the construction of  $L$  is a main difficulty in the paper.

We use a special choice of a function  $\eta(x, y) = U_N(y) + \gamma u_1(x, y)$  that determines the heat source, where  $\gamma > 0$  is a small parameter. This parameter defines the nonlinearity, i.e., the IBVP is weakly nonlinear for small  $\gamma$ . In  $\eta(x, y)$  the argument  $x$  is a horizontal space variable,  $y$  is a vertical axis and the Marangoni conditions hold at the boundary  $y = 0$ . The terms  $U_N(y)$  and  $\gamma u_1$  play different roles. The operator  $L$  is defined by  $U_N$  and it does not depend on  $\gamma$ . The function  $u_1$  defines the matrix  $M$  in equations (1.1).

To simplify analysis, we set periodic boundary conditions along  $x$ . Then, since  $U_N$  depends only on  $y$ , we can separate variables in the spectral problem for the linear operator  $L$  (this method is well known, see [27, 9, 11, 2, 30]). Eigenfunctions of  $L = L_N$  have the form  $\exp(ikx)\theta_k(y)$  with eigenvalues  $\lambda_k$ , where  $k = 1, 2, \dots$ . For each  $k$  we reduce the spectral problem to a nonlinear equation for the spectral parameter  $\lambda_k$ . For a special choice of the function  $U_N$  this equation can be investigated and we can check explicitly that  $L_N$  has a spectral barrier (gap), mentioned above. The investigation of this nontrivial nonlinear equation is the most complicated part of the paper.

Note that the choice of  $U_N(y)$  admit a transparent physical interpretation. The function  $U_N(y)$  consists of two terms,  $U_N = H(y, \mu) + \mu W_N(y, \mu)$ , where  $\mu$  is a second small parameter (independent of  $\nu$  and  $\gamma$ ). For small  $\mu$  the first term  $H_N$  is close to a step function, where the step is located at the point  $y_0(\mu)$ , which lies at the boundary  $y = 0$ , where the Marangoni condition holds. This means that we have "a heat shock" at the boundary. The second term  $\mu W_N$  is a small polynomial perturbation of  $H$ . For  $U_N = H$  we have the operator with the eigenvalues  $\lambda_k$ , which are close to zero for bounded  $k$ . A small polynomial term  $\mu W_N$  perturbs the spectrum as follows. For an especial choice of  $\mu W_N$  we have  $\lambda_k = 0$ ,  $k = 1, 2, \dots, N$  whereas all others  $\lambda_k$  satisfy  $\text{Re } \lambda_k < -\delta(\mu)$ , where  $\delta > 0$ . This situation is

similar to the classical Rayleigh- Bénard convection [27, 9, 11], where  $N = 1$  and  $\lambda_k = 0$  for a single wave vector  $k = k_0$  only. A key difference with respect to this well known case is that our special construction for the Marangoni-Bénard case produces a number of  $k$  with  $\lambda_k = 0$ , and an interaction of the corresponding slow modes can generate a complicated dynamics.

The gap property of the operator  $L_N$  allows us to proceed the reduction of the NS equations to (1.1) by a quite routine procedure, which uses the well known results of invariant manifold theory [13, 1, 4, 3]. This procedure shows that  $K$  and  $M$  depends on the source  $\eta$  in a special way, namely, the coefficients involved in  $K$  depend on the eigenfunctions of  $L = L_N$  whereas  $M$  is a linear functional of correction  $u_1$ . We show that the range of this functional is dense in the linear space of all  $N \times N$ -matrices. This fact allows us to apply our results on quadratic systems (1.1) from section 8 and completes the proof.

The paper is organized as follows. In the next section we formulate the Marangoni problem. In Section 3 we state the main result. Section 4 describes the RVF method for weakly nonlinear systems. In Section 5 it is shown that the Marangoni problem is well posed and defines a local semiflow. In Section 6 we investigate a linear operator describing a linearization of our IBVP, and show that this operator has a spectral gap. In Section 7 we prove existence of the finite dimensional invariant manifold. In Section 8 we consider quadratic systems (1.1). In Section 9 we check conditions, which is critically important for the RVF method. Here we show that, for each fixed  $N$ , by a choice  $u_1(x, y)$ , we can obtain any prescribed matrix  $\mathbf{M}$ . The complete algorithm of the RVF method uses this fact and it is stated in Section 10.

Below we use the following standard convention: all positive constants, independent of small parameters  $\epsilon, \gamma, \dots$ , are denoted by  $c_i, C_j$ . To diminish a formidable number of indices  $i, j$ , we shall use sometimes the same indices assuming that the constants can vary from a line to a line.

## 2. Marangoni problem for Navier Stokes equations

We consider the Navier Stokes system for an ideal incompressible fluid

$$(2.2) \quad \mathbf{v}_t + (\mathbf{v} \cdot \nabla) \mathbf{v} = \nu \Delta \mathbf{v} - \nabla p,$$

$$(2.3) \quad \nabla \cdot \mathbf{v} = 0,$$

$$(2.4) \quad u_t + (\mathbf{v} \cdot \nabla) u = \Delta u + \eta,$$

where  $\mathbf{v} = (v_1(x, y, t), v_2(x, y, t))^{tr}$ ,  $u = u(x, y, t)$ ,  $p = p(x, y, t)$  are unknown functions defined on  $\Omega \times \{t \geq 0\}$ ,  $\Omega$  is the strip  $(-\infty, \infty) \times [0, h] \subset \mathbf{R}^2$ . Here  $\mathbf{v}$  is the fluid velocity, where  $v_1$  and  $v_2$  are the normal and tangent velocity components,  $\nu$  is the viscosity coefficient,  $p$  is the pressure,  $u$  is the temperature,  $\eta(x, y)$  is a function describing a distributed heat source,  $\mathbf{v} \cdot \nabla$  denotes the advection operator  $v_1 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y}$ . Notice that since in (2.4) the thermal diffusion rate equals 1, the viscosity coefficient  $\nu$  can be identified with the Prandtl number.

The initial conditions are

$$(2.5) \quad \mathbf{v}(x, y, 0) = \mathbf{v}^0(x, y), \quad p(x, y, 0) = p^0(x, y), \quad u(x, y, 0) = u^0(x, y).$$

Let us suppose that the unknown functions are  $2\pi$ -periodic in  $x$ :

$$(2.6) \quad \mathbf{v}(x, y, t) = \mathbf{v}(x + 2\pi, y, t), \quad p(x, y, t) = p(x + 2\pi, y, t),$$

$$(2.7) \quad u(x, y, t) = u(x + 2\pi, y, t),$$

and that  $u^0, p^0, \mathbf{v}^0$  also are  $2\pi$ -periodic in  $x$ . The function  $u$  satisfies the Neumann boundary conditions:

$$(2.8) \quad u_y(x, y, t)|_{y=h} = 0, \quad u_y(x, y, t)|_{y=0} = 0.$$

We assume that the surface  $y = h$  is free:

$$(2.9) \quad v_2(x, h, t) = 0, \quad \frac{\partial v_1(x, y, t)}{\partial y}|_{y=h} = 0.$$

The Marangoni boundary condition at  $y = 0$  is defined by a relation connecting the tangent velocity component and the tangent gradient of the temperature:

$$(2.10) \quad v_{1y}(x, y, t)|_{y=0} = -\gamma_0 u_x(x, 0, t),$$

where  $\gamma_0 > 0$  is a coefficient (the Marangoni parameter). We set below  $\gamma_0 = 1$  to simplify formulas. For  $v_2$  at  $y = 0$  one has

$$(2.11) \quad v_2(x, 0, t) = 0.$$

Let us assume that

$$(2.12) \quad \langle \eta, 1 \rangle = \int_{\Omega} \eta(x, y) dx dy = 0,$$

where  $\langle u, v \rangle$  is the scalar product in  $L_2(\Omega)$ :

$$(2.13) \quad \langle u, v \rangle = \int_0^h \left( \int_0^{2\pi} u(x, y) v(x, y) dx \right) dy.$$

Note that if  $u(x, y, t)$  is a solution to (2.4), (2.8) and (2.7), then for any constant  $C$  the function  $u(x, y, t) + C$  also is a solution.

We use below the stream function - vorticity formulation of these equations in order to exclude the pressure  $p$ . Introducing the vorticity  $\omega$  and the stream function  $\psi$ , we obtain [5]

$$(2.14) \quad \Delta \psi = -\omega,$$

where the velocity  $\mathbf{v}$  can be expressed through the stream function  $\psi(x, y)$  by the relations  $v_1 = \psi_y, v_2 = -\psi_x$ . Equations (2.2), (2.3) and (2.4) take the form [5]

$$(2.15) \quad \omega_t + \{\psi, \omega\} = \nu \Delta \omega,$$

here  $\{\psi, \omega\} = \psi_y \omega_x - \psi_x \omega_y$ ,

$$(2.16) \quad u_t + \{\psi, u\} = \Delta u + \eta.$$

The boundary conditions become

$$(2.17) \quad \psi(x, y, t) = \psi(x + 2\pi, y, t), \quad \omega(x, y, t) = \omega(x + 2\pi, y, t),$$

$$(2.18) \quad \psi(x, y, t)|_{y=h} = \omega(x, y, t)|_{y=h} = 0,$$

$$(2.19) \quad \psi(x, y, t)|_{y=0} = 0, \quad \omega(x, y, t)|_{y=0} = u_x(x, 0, t).$$

$$(2.20) \quad u_y(x, y, t)|_{y=h} = 0, \quad u_y(x, y, t)|_{y=0} = 0.$$

The vortex-stream reformulation of the problem is given by equations and boundary conditions (2.14) - (2.20) and initial conditions

$$(2.21) \quad u(x, y, 0) = u_0(x, y), \quad \omega(x, y, 0) = \omega_0(x, y).$$

### 3. Main result

Before to formulate the main theorem, let us describe the method of the realization of vector fields (RVF) invented by P. Poláčik (see [25, 26]). We change slightly the original version to adapt it for our goals.

Let us consider a family of local semiflows  $S_{\mathcal{P}}^t$  in a fixed Banach space  $B$ . Assume these semiflows depend on a parameter  $\mathcal{P} \in B_1$ , where  $B_1$  is another Banach space. Denote by  $\mathcal{B}^n$  the unit ball  $\{q : |q| \leq 1\}$  in  $\mathbf{R}^n$ , where  $q = (q_1, q_2, \dots, q_n)$  and  $|q|^2 = q_1^2 + \dots + q_n^2$ . Remind that a set  $M$  is said to be locally invariant in an open set  $W \subset B$  under a semiflow  $S_t$  in  $B$  if  $M$  is a subset of  $W$  and each trajectories of  $S_t$  leaving  $M$  simultaneously leaves  $W$ . Consider a system of differential equations defined on the ball  $\mathcal{B}^n$ :

$$(3.22) \quad \frac{dq}{dt} = Q(q),$$

where

$$(3.23) \quad Q \in C^1(\mathcal{B}^n), \quad \sup_{q \in \mathcal{B}^n} |\nabla Q(q)| < 1.$$

Assume the vector field  $Q$  is directed strictly inward at the boundary  $\partial\mathcal{B}^n = \{q : |q| = 1\}$ :

$$(3.24) \quad Q(q) \cdot q < 0, \quad q \in \partial\mathcal{B}^n.$$

Then system (3.22) defines a global semiflow on  $\mathcal{B}^n$ . Let  $\epsilon$  be a positive number.

**DEFINITION 3.1. (realization of vector fields)** *We say that the family of local semiflows  $S_{\mathcal{P}}^t$  realizes the vector field  $Q$  (dynamics (3.22)) with accuracy  $\epsilon$  (briefly,  $\epsilon$  - realizes), if there exists a parameter  $\mathcal{P} = \mathcal{P}(Q, \epsilon, n) \in B_1$  such that*

- (i) *semiflow  $S_{\mathcal{P}}^t$  has a positively invariant and locally attracting manifold  $\mathcal{M}_n \subset B$  diffeomorphic to the unit ball  $\mathcal{B}^n$ ;*
- (ii) *this manifold is embedded into  $B$  by a map*

$$(3.25) \quad z = Z(q), \quad q \in \mathcal{B}^n, \quad z \in B, \quad Z \in C^{1+r}(\mathcal{B}^n),$$

where  $r > 0$ ;

- (iii) *the restriction of the semiflow  $S_{\mathcal{P}}^t$  to  $\mathcal{M}_n$  is defined by the system of differential equations*

$$(3.26) \quad \frac{dq}{dt} = Q(q) + \tilde{Q}(q, \mathcal{P}), \quad Q \in C^1(\mathcal{B}^n),$$

where

$$(3.27) \quad |\tilde{Q}(\cdot, \mathcal{P})|_{C^1(\mathcal{B}^n)} < \epsilon.$$

This means that the dynamics on the invariant manifold is defined by the variables  $q_1, q_2, \dots, q_n$  and approximates prescribed dynamics (3.22) with accuracy  $\epsilon$ .

The IBVP defined by (2.14) -(2.21) involves the numbers  $\nu, h$  and the function  $\eta(x, y)$ . We set  $\mathcal{P} = \{h, \nu, \eta(\cdot, \cdot)\}$ . The main result is as follows:

**THEOREM 3.2.** *Dynamics of the semiflows defined by IBVP (2.14) -(2.21) , is maximally complex in the following sense. For each integer  $n$ , each  $\epsilon > 0$  and each vector field  $Q$  satisfying (3.23) and (3.24), there exists a value of the parameter  $\mathcal{P}(Q, \epsilon)$  of this IBVP such that this problem defines a semiflow  $S_{\mathcal{P}}^t$ , which  $\epsilon$ -realizes the vector field  $Q$ .*

This result implies the following corollary.

**Corollary.** *The family of semiflows  $S^t(\mathcal{P})$  induced by IBVP (2.14) -(2.21) with parameter  $\mathcal{P}$ , generate all (up to orbital topological equivalencies) hyperbolic dynamics on compact invariant hyperbolic sets defined by  $C^1$ -smooth vector fields on finite dimensional smooth compact manifolds.*

In particular, we find that the Navier-Stokes dynamics can generate Smale axiom A flows, Ruelle-Takens attractors [29, 22], the Anosov flows, and, due to the persistence of compact invariant hyperbolic sets [28, 15], hyperbolic dynamics defined on these sets.

#### 4. RVF method for weakly nonlinear evolution equations

This section describes a general construction of the RVF method for "small" solutions. Let us consider an evolution equation

$$(4.28) \quad v_t = Lv + F(v) + \gamma f,$$

where  $v$  lies in an Hilbert space  $H$ ,  $L$  is a sectorial operator,  $F$  is a nonlinear operator,  $f \in H$  is independent of  $v, t$  ("an external force") and  $\gamma > 0$  is a small parameter. We use the standard function spaces [13]

$$H_\alpha = \{v \in H : \|v\|_\alpha = \|(I - L)^\alpha v\| < \infty\}.$$

Assume  $F$  is a  $C^{1+r}$  map from  $H_\alpha$  to  $H$ ,  $r \in (0, 1)$ . We set

$$(4.29) \quad v(0) = v_0, \quad v_0 \in H_\alpha.$$

We also suppose that this map satisfies conditions

$$(4.30) \quad \|F(v)\| \leq C_1 \|v\|_\alpha^2, \quad \|DF(v)\| \leq C_2 \|v\|_\alpha$$

for some  $\alpha \in (0, 1)$ .

Then a unique solution of the Cauchy problem (4.28), (4.29) exists on some open time interval  $(0, t_0(v_0))$ ,  $t_0 > 0$  [13]. The following assumption plays a key role.

**Spectral Gap Condition.** *Assume the spectrum  $\text{Spec } L \subset \mathbf{C}$  of  $L$  consists of the two parts:  $\text{Spec } L = \{0\} \cup \mathcal{S}$ , where*

$$(4.31) \quad \text{Re } z < -c_0 < 0 \quad \text{for all } z \in \mathcal{S}$$

*and there exist exactly  $N$  linearly independent  $e_j \in H$  such that*

$$(4.32) \quad Le_j = 0, \quad j = 1, \dots, N.$$

Let  $B_1$  be the space  $B_1 = \text{Span}\{e_1, \dots, e_N\}$ . Then there exists a space  $B_2$  invariant under  $L$  such that  $H = B_1 + B_2$ , where  $B_1 + B_2$  is a direct sum of  $B_i$  ([6, 7], also see [13], Th. 1.5.2). We have two complementary projection operators  $\mathbf{P}_1$  and  $\mathbf{P}_2$  such that  $\mathbf{P}_1 + \mathbf{P}_2 = \mathbf{I}$ , where  $\mathbf{I}$  denotes the identity operator, and  $B_i = \mathbf{P}_i H$ . Let us denote by  $L^*$  an operator conjugate to  $L$ . If the operator  $L$  has a compact resolvent, then the spectra of  $L$  and  $L^*$  are discrete and we have countable sets of eigenvectors  $e_j$  and  $\bar{e}_j$  of the operators  $L$  and  $L^*$  respectively,  $j \in \mathbf{N} = \{1, 2, \dots\}$ . In this case without loss of generality one can assume that  $e_i$

and  $\tilde{e}_i$  are biorthogonal:  $\langle e_i, \tilde{e}_j \rangle = \delta_{ij}$ , where  $\delta_{ij}$  stands for the Kronecker symbol. Then  $P_1$  can be defined by

$$(4.33) \quad \mathbf{P}_1 u = \sum_{i=1}^N \langle u, \tilde{e}_i \rangle e_i,$$

where  $L^* \tilde{e}_i = 0$ ,  $i = 1, \dots, N$ .

Consider small solutions of (4.28) of the following form:

$$(4.34) \quad v = \gamma X + w, \quad w(t) \in B_2, \quad X(t) = \sum_{i=1}^N X_i(t) e_i \in B_1.$$

Substituting (4.34) to eq. (4.28) we obtain

$$(4.35) \quad X_t = \gamma^{-1} \mathbf{P}_1 F(\gamma X + w) + f_1,$$

$$(4.36) \quad w_t = Lw + \mathbf{P}_2 F(\gamma X + w) + \gamma f_2,$$

where  $f_k = \mathbf{P}_k f$ . Assume

$$f_1 = \gamma \bar{f}_1, \quad \|\bar{f}_1\| < \bar{C}_1, \quad \|f_2\| < C_2.$$

Let us set

$$w = \gamma w_0 + \tilde{w}$$

where  $w_0$  is defined by

$$w_0 = -L^{-1} f_2.$$

We consider (4.35), (4.36) in the domain

$$(4.37) \quad \mathcal{D}_{R_1, \gamma, C} = \{(X, \tilde{w}) : |X| < R_1, \|\tilde{w}\|_\alpha < C\gamma^2\}.$$

Let  $\mathcal{B}^N(R_0)$  be the ball  $\mathcal{B}^N(R_0) = \{X : |X| < R_0\}$ . The following assertion will be useful below.

**LEMMA 4.1.** *Assume  $r \in (0, 1)$ ,  $\alpha \in (0, 1)$  and  $C > C_0(R_1, F, \alpha)$  is large enough. Then for sufficiently small positive  $\gamma < \gamma_0(r, \alpha, F, C, R, N)$  system (4.35), (4.36) has a locally invariant in  $\mathcal{D}_{R_1, \gamma, C}$  and locally attracting manifold  $\mathcal{M}_{N, \gamma}$  defined by*

$$(4.38) \quad w = \gamma(w_0 + \tilde{W}(X, \gamma)),$$

where a  $C^{1+r}$  smooth map  $\tilde{W} : \mathcal{B}^N(R) \rightarrow H_\alpha$  satisfies the estimates

$$(4.39) \quad \sup_{X \in \mathcal{B}^N(R)} \|\tilde{W}(\cdot, \gamma)\|_\alpha + \sup_{X \in \mathcal{B}^N(R)} \|D_X \tilde{W}(\cdot, \gamma)\|_\alpha < c_1 \gamma.$$

**Proof.** This assertion is an immediate consequence of Theorem 6.1.7 [13]. The proof is standard and can be found in Appendix 2.

On the manifold  $\mathcal{M}_{N, \gamma}$  evolution equation (4.35) for the slow component  $X$  takes the following form:

$$(4.40) \quad \frac{dX}{d\tau} = K(X, \gamma) + M(\gamma)X + \hat{f}_1 + \phi(X, \gamma),$$

where  $\tau = \gamma t$ ,

$$(4.41) \quad K(X, \gamma) = \mathbf{P}_1 \gamma^{-2} (F(\gamma(X + w_0)) - \gamma DF(\gamma w_0)X - F(\gamma w_0)),$$

and  $M(\gamma) : \mathbf{R}^N \rightarrow \mathbf{R}^N$  is a bounded linear operator defined by

$$(4.42) \quad M(\gamma)X = \mathbf{P}_1 \gamma^{-1} DF(\gamma w_0)X.$$



We have

$$(4.43) \quad \hat{f}_1 = \bar{f}_1 + \mathbf{P}_1 \gamma^{-2} F(\gamma w_0),$$

and  $\phi$  is a small correction such that

$$(4.44) \quad |\phi|_{C^{1+r}(\mathcal{B}^N(R))} < c_5 \gamma, \quad r > 0.$$

For quadratic nonlinearities  $F$  such that

$$F(\alpha v) = \alpha^2 F(v)$$

the relations for  $K$  and  $M$  can be simplified ( $K$  and  $M$  do not depend on  $\gamma$ ):

$$(4.45) \quad K(X) = \mathbf{P}_1(F(X + w_0) - DF(w_0)X - F(w_0)),$$

$$(4.46) \quad M(w_0)X = \mathbf{P}_1 DF(w_0)X.$$

The key idea is to consider the operator  $M$  as a parameter in the RVF method. This idea works if the following property holds.

**Linear operator density (LOD) condition.** *Let us consider the set  $\mathbf{O}_F$  of all linear operators  $\mathbf{M}(L^{-1}f)$  that can be obtained by (4.46) when  $f$  runs over the whole space  $B_2$ . We assume that this set  $\mathbf{O}_F$  is dense in the set of all linear operators  $\mathbf{R}^N \rightarrow \mathbf{R}^N$ .*

In coming sections we apply this general approach to IBVP (2.14)-(2.21).

## 5. Existence and uniqueness for IBVP (2.14)-(2.21) and auxiliary estimates.

**5.1. Function spaces and embeddings.** We use standard Hilbert spaces [13]. We denote by  $H = L_2(\Omega)$  the Hilbert space of measurable,  $2\pi$ -periodical in  $x$  functions defined on  $\Omega$  with bounded norms  $\|\cdot\|$ , where  $\|u\|^2 = \langle u, u \rangle$  and  $\langle \cdot, \cdot \rangle$  is the inner product defined by (2.13). Let us denote by  $H_\alpha$  the fractional spaces

$$(5.47) \quad H_\alpha = \{\omega : \|\omega\|_\alpha = \|(I - \Delta_D)^\alpha \omega\| < \infty\},$$

here  $\Delta_D$  is the Laplace operator with the standard domain corresponding to the zero Dirichlet boundary conditions:

$$(5.48) \quad \text{Dom } \Delta_D = \{\omega : \omega \in W_{2,2}(\Omega), \quad \omega(x, y)|_{y=0, y=h} = 0\},$$

here  $W_{q,2}(\Omega)$  denote the standard Sobolev spaces. Let  $\tilde{H}_\alpha$  be another fractional space associated with  $L_2(\Omega)$ :

$$(5.49) \quad \tilde{H}_\alpha = \{u : \|u\|_\alpha = \|(I - \Delta_N)^\alpha u\| < \infty\},$$

where  $\Delta_N$  is the Laplace operator with the domain corresponding to the zero Neumann boundary conditions

$$(5.50) \quad \text{Dom } \Delta_N = \{u : u \in W_{4,2}(\Omega), \quad u_y(x, y)|_{y=0, y=h} = 0\}.$$

We omit sometimes the indices  $N, D$ . This choice of the domain for  $\Delta_N$  is connected with a special choice of main function space for  $u$ -component. The Sobolev embeddings

$$(5.51) \quad H_\alpha \subset C^s(\Omega), \quad 0 \leq s < 2(\alpha - 1/2),$$

and

$$(5.52) \quad H_\alpha \subset L_q(\Omega), \quad 1/q > 1/2 - \alpha, \quad q \geq 2$$

are useful below, the same embeddings hold for  $\tilde{H}_\alpha$ . Let  $Tr(u)$  be the trace of a function  $u$  on the bottom boundary  $S$  of  $\Omega$  (where  $y = 0$ ). We shall use the following embedding [31]:

$$(5.53) \quad \|Tr(u)\|_{L_2(S)} \leq c\|u\|_\alpha, \quad \alpha > 1/4.$$

**5.2. Some preliminaries and auxiliary estimates.** In coming subsections, our aim is to prove that IBVP (2.14)-(2.21), defines a local semiflow. Moreover, we need some estimates important for the invariant manifold technique. To show existence of solutions we use the standard semigroup methods. Here, however, we met some difficulties because the Marangoni condition induces a singularity [23]. To circumvent them, we choose an appropriate phase space taking into account that the temperature field  $u$  should be more regular than the vorticity field  $\omega$ . Moreover, we use a special representation of the vorticity in order to represent IBVP (2.14)-(2.21) by evolution equations.

Let us consider IBVP (2.14)-(2.21) in the phase space

$$(5.54) \quad \mathcal{H} = H \times \tilde{H}_1,$$

i.e., we use the space  $\tilde{H}_1$  for  $u$ -component and the space  $H$  for  $\omega$ -component.

In order to apply semigroup technique, we represent  $\omega$  as a sum of two terms,  $\omega = \bar{\omega} + \tilde{\omega}$ , where the second term  $\tilde{\omega}$  satisfies the zero Dirichlet boundary conditions, and  $\bar{\omega}$  is defined as a solution of the following linear boundary value problem:

$$(5.55) \quad \Delta \bar{\omega} = 0,$$

$$(5.56) \quad \bar{\omega}(x, 0, t) = u_x(x, 0, t), \quad \bar{\omega}(x, h, t) = 0.$$

We have  $\psi = \bar{\psi} + \tilde{\psi}$ , where the functions  $\bar{\psi}$  and  $\tilde{\psi}$  are defined as solutions of the boundary value problems

$$(5.57) \quad \Delta \bar{\psi} = -\bar{\omega},$$

$$(5.58) \quad \bar{\psi}(x, h, t) = \bar{\psi}(x, 0, t) = 0,$$

$$(5.59) \quad \Delta \tilde{\psi} = -\tilde{\omega},$$

$$(5.60) \quad \tilde{\psi}(x, h, t) = \tilde{\psi}(x, 0, t) = 0.$$

These problems can be resolved by the Fourier series. It is clear that, for sufficiently smooth  $u$ , boundary value problem (5.55),(5.56) defines a linear operator  $u \rightarrow \bar{\omega}(u)$ . The following lemma gives useful estimates of this operator.

LEMMA 5.1. *The map  $u \rightarrow \bar{\omega}(u)$  satisfies*

$$(5.61) \quad \|\bar{\omega}(u)\| \leq c_1\|u\|_\alpha, \quad \alpha > 1/2,$$

$$(5.62) \quad \|\nabla \bar{\omega}(u)\| \leq \|\bar{\omega}\|_{1/2} \leq c_2\|u\|_{\alpha_1}, \quad \alpha_1 > 1,$$

and for solutions  $\bar{\psi}$  of (5.59), (5.60) one has

$$(5.63) \quad \sup |\nabla \bar{\psi}| \leq c_3\|u\|_\alpha, \quad \|\nabla \bar{\psi}\| \leq c_3\|u\|_\alpha, \quad \alpha > 3/2.$$

We prove the lemma using explicit solutions of boundary value problems (5.55), (5.56) and (5.57), (5.58) by the Fourier series. The function  $\bar{\omega}$  can be expressed via the Fourier coefficients of the trace  $u$  on  $S$  (see Appendix 1, (13.291)). This gives

$$\|\bar{\omega}\| \leq c_1 \|Tr((I - D_x^2)^{1/4} u)\|_{L_2(S)}, \quad D_x = \frac{\partial}{\partial x}.$$

By (5.53) one obtains (5.61). Similarly,

$$\|\nabla \bar{\omega}\| \leq c \|Tr((I - D_x^2)^{3/4} u)\|_{L_2(S)},$$

and embedding (5.53) implies (5.62). Estimate (5.63) also follows from the Fourier series (see Appendix 1).

**5.3. Evolution equations for  $\bar{\omega}$  and  $u$ .** Let us introduce some auxiliary maps. Let  $\psi(\omega(\cdot, \cdot, \cdot))$  be a linear functional of  $\omega$  defined as a periodic in  $x$  solution of the boundary problem

$$(5.64) \quad \Delta \psi = -\omega, \quad \psi(x, 0, t) = \psi(x, h, t) = 0.$$

Below to simplify notation we denote  $\psi(\omega(\cdot, \cdot, \cdot))$  simply  $\psi(\omega)$  or  $\psi$ .

The map  $G : \omega \rightarrow G(\omega)$  is defined on  $H$  by

$$(5.65) \quad G(\omega) = -\{\psi(\omega), \omega\}.$$

The map  $F$  is defined on  $H \times \tilde{H}_1$  by

$$(5.66) \quad F(\omega, u) = -\{\psi(\omega), u\}.$$

Moreover, we introduce

$$(5.67) \quad Z(\bar{\omega}, u) = G(\bar{\omega} + \bar{\omega}(u)) - \bar{\omega}(F).$$

One has

$$\bar{\omega}(u_t) = \bar{\omega}(\Delta u + F(\bar{\omega} + \bar{\omega}, u)).$$

Using this relation and definitions (5.65), (5.66) and (5.67) we rewrite IBVP defined by (2.14) - (2.21) as a system of evolution equations:

$$(5.68) \quad u_t = L_2 u + F(\bar{\omega} + \bar{\omega}(u), u) + \eta,$$

$$(5.69) \quad \bar{\omega}_t = L_1 \bar{\omega} + Z(\bar{\omega}, u),$$

where  $L_2 = \Delta_N$  is the Laplace operator under the zero Neumann boundary conditions (see (5.50)), and the linear operator  $L_1$  is defined by

$$(5.70) \quad L_1 \bar{\omega} = \nu \Delta_D \bar{\omega} - \bar{\omega}(\Delta_N u).$$

Note that boundary conditions (2.8) and others are taken into account by domain definitions for operators  $\Delta_D$  and  $\Delta_N$ .

**5.4. Smoothness of maps  $F$  and  $Z$ .** We define spaces  $\mathcal{H}_\alpha$  with  $\alpha \geq 0$  and  $\mathcal{H} = \mathcal{H}_0$  by

$$(5.71) \quad \mathcal{H}_\alpha = H_\alpha \times \tilde{H}_{\alpha+1}, \quad \mathcal{H} = H \times \tilde{H}_1.$$

In order to apply the semigroup approach [13] to system (5.68, 5.69), let us show first that  $F : (\bar{\omega}, w) \rightarrow F(\bar{\omega}, w)$  and  $Z : (\bar{\omega}, w) \rightarrow Z(\bar{\omega}, w)$  are  $C^1$  smooth maps from the space  $\mathcal{H}_\alpha$  to the spaces  $\tilde{H}_1$  and  $H$ , respectively.

LEMMA 5.2.  *$F$  defines a bounded map from  $\mathcal{H}_\alpha$  to  $\tilde{H}_1$ .*

To prove this assertion, we use the estimate

$$(5.72) \quad \|F(\tilde{\omega}, u)\|_1 \leq S_1 + S_2 + S_3 + S_4,$$

where

$$\begin{aligned} S_1 &= \|\{(I - \Delta)\tilde{\psi}, u\}\|, \quad S_2 = 2\|\nabla\tilde{\psi}_x \cdot \nabla u_y - \nabla\tilde{\psi}_y \cdot \nabla u_x\|, \\ S_3 &= \|\{\tilde{\psi}, \Delta u\}\|, \quad S_4 = \|\{\tilde{\psi}, u\}\|. \end{aligned}$$

By embeddings (5.51) one has

$$S_1 \leq c_1 \|\nabla\Delta\tilde{\psi}\| \sup |\nabla u| \leq c_1 \|\tilde{\omega}\|_{1/2} \|u\|_{\alpha+1}, \quad \alpha > 0.$$

In a similar way,

$$S_2 \leq c_2 (\|\nabla\tilde{\psi}_x\| + \|\nabla\tilde{\psi}_y\|) \|u\|_{C^2} \leq c_3 \|\tilde{\omega}\| \|u\|_{\alpha+1}, \quad \alpha > 1/2.$$

To estimate  $S_3$  we use the inequalities

$$S_3 \leq c_4 \|\nabla\Delta u\| \sup |\nabla\tilde{\psi}| \leq c_5 \|u\|_{3/2} \|\tilde{\omega}\|_{\alpha}, \quad \alpha > 1/2.$$

For  $S_4$  one has

$$S_4 \leq 2\|\nabla\tilde{\psi}\| \sup |\nabla u| \leq c_6 \|\tilde{\omega}\| \|u\|_{\alpha}.$$

Therefore,

$$(5.73) \quad \|F(\tilde{\omega}, u)\|_1 \leq c \|\tilde{\omega}\|_{\alpha} \|u\|_{1+\alpha}, \quad \alpha \in (1/2, 1).$$

Let us estimate  $\|F(\bar{\omega}, u)\|_1$ . Again one has

$$(5.74) \quad \|F(\bar{\omega}, u)\|_1 = \bar{S}_1 + \bar{S}_2 + \bar{S}_3 + \bar{S}_4,$$

where

$$\begin{aligned} \bar{S}_1 &= \|\{\bar{\psi} - \Delta\bar{\psi}, u\}\|, \quad \bar{S}_2 = 2\|\nabla\bar{\psi}_x \cdot \nabla u_y - \nabla\bar{\psi}_y \cdot \nabla u_x\|, \\ \bar{S}_3 &= \|\{\bar{\psi}, \Delta u\}\|, \quad \bar{S}_4 = \|\{\bar{\psi}, u\}\|. \end{aligned}$$

Repeating the same arguments as above and using Lemma 5.1, one has

$$\bar{S}_1 \leq c_1 (\|\nabla\Delta\bar{\psi}\| + \|\nabla\bar{\psi}\|) \sup |\nabla u| \leq c_2 \|\bar{\omega}\|_{1/2} \|u\|_{\alpha+1} \leq c_3 \|u\|_{\alpha+1}^2, \quad \alpha > 0.$$

In a similar way, one obtains

$$\bar{S}_2 \leq c_4 (\|\nabla\bar{\psi}_x\| + \|\nabla\bar{\psi}_y\|) \|u\|_{C^2} \leq c_5 \|\bar{\omega}\| \|u\|_{\alpha+1} \leq c_6 \|u\|_{\alpha+1}^2, \quad \alpha > 1/2.$$

Moreover,

$$\bar{S}_3 \leq c_7 \|\nabla\Delta u\| \sup |\nabla\bar{\psi}| \leq c_7 \|u\|_{3/2} \|\bar{\omega}\|_{\alpha} \leq c_8 \|u\|_{\alpha+1}^2, \quad \alpha > 1/2$$

and

$$\bar{S}_4 \leq 2\|\nabla\bar{\psi}\| \sup |\nabla u| \leq c_9 \|\bar{\omega}\| \|u\|_{\alpha+1} \leq c_{10} \|u\|_{\alpha+1}, \quad \alpha > 1/2.$$

Therefore,

$$(5.75) \quad \|F(\bar{\omega}, u)\|_1 \leq c_7 \|u\|_{\alpha+1}^2, \quad \alpha > 1/2.$$

Combining this estimate with (5.73) one concludes that  $F$  is a bounded map from  $\mathcal{H}_{\alpha}$  to  $\tilde{H}_1$ .

LEMMA 5.3.  $Z$  defines a bounded map from  $\mathcal{H}_{\alpha}$  to  $H$ .

To prove this lemma, let us notice

$$(5.76) \quad ||\{\psi, \omega\}|| \leq c_1(\sup |\nabla \tilde{\psi}| + \sup |\nabla \bar{\psi}|)(||\tilde{\omega}||_{1/2} + ||\bar{\omega}||_{1/2}).$$

This gives, by (5.63), Lemma 5.1, and the Sobolev embeddings (5.51) that

$$(5.77) \quad ||\{\psi, \omega\}|| \leq c_4(||u||_{\alpha+1}^2 + ||\tilde{\omega}||_{\alpha}^2), \quad \alpha > 1/2.$$

Let us estimate  $\bar{\omega}(F)$ . By Lemma 5.1 one has

$$(5.78) \quad ||\bar{\omega}(F)|| \leq c_1||F||_{\gamma}, \quad \gamma > 1/2.$$

One has  $||F||_{\gamma} \leq c_2||F||_1$ . The estimate of  $||F||_1$  follows from Lemma 5.2. Therefore,

$$(5.79) \quad ||\bar{\omega}(F)|| \leq c_3(||u||_{\alpha+1}^2 + ||\tilde{\omega}||_{\alpha}^2), \quad \alpha > 1/2.$$

The proof is complete.

Lemmas 5.2 and 5.3 show that  $F$  and  $Z$  are bounded maps from a bounded domain in  $\mathcal{H}_{\alpha}$  to  $\mathcal{H}$ . They are quadratic and analogous estimates imply that the derivatives of  $F, G$  are bounded as maps from  $\mathcal{H}_{\alpha}$  to  $\mathcal{H}$ . Indeed, the derivative of  $D_{\tilde{\omega}}G$  with respect to  $\tilde{\omega}$  is a linear operator from  $H_{\alpha}$  to  $H$  defined by

$$(D_{\omega}G)\delta\tilde{\omega} = \{\delta\tilde{\psi}, \omega\} + \{\psi, \delta\tilde{\omega}\}.$$

An estimate of the norm of this operator can be found as above.

**5.5. Transformation of evolution equations (5.68), (5.69) and associated linear operator.** We follow the standard approach developed for the Rayleigh- Bénard and Marangoni -Bénard convection [9, 11, 2]. Assume that the temperature field  $u$  is a small  $\gamma$  -perturbation of a vertical profile  $U(y)$ . Here  $\gamma > 0$  is a small parameter independent of the viscosity  $\nu$  (this assumption is important):  $\gamma < \gamma_0(\nu)$ . Let  $U$  be a  $C^{\infty}$ -smooth function of  $y \in [0, h]$  such that

$$(5.80) \quad U(y) = U_y(y) = 0, \quad \forall y \in [0, \delta_1),$$

for some  $\delta_1 \in (0, h)$ . Assume  $u_1(x, y)$  is a  $C^{\infty}$  smooth  $2\pi$  periodical in  $x$  function satisfying conditions

$$(5.81) \quad u_1(x, y) = 0 \quad \forall x \in (-\infty, +\infty), \quad \forall y \in (\delta_1, h].$$

We set

$$u_0 = U + \gamma u_1$$

and

$$\begin{aligned} \eta &= \eta_0 + \gamma^2 \eta_1, \quad \eta_0 = -\Delta u_0, \\ \langle \eta_1, 1 \rangle &= 0, \quad |\eta_1|_{C^3(\Omega)} < C_0. \end{aligned}$$

Let us represent  $u$  as

$$(5.82) \quad u = u_0 + w,$$

where  $w$  a new unknown function.

For new unknowns  $\tilde{\omega}$ ,  $w$  system (5.68), (5.69) takes the form

$$(5.83) \quad \tilde{\omega}_t = \nu \Delta \tilde{\omega} - \bar{\omega}(w_t) - \{\psi, \bar{\omega} + \tilde{\omega}\},$$

$$(5.84) \quad w_t = \Delta w - \{\psi, U + \gamma u_1 + w\} + \gamma^2 \eta_1,$$

where

$$(5.85) \quad \omega = \bar{\omega} + \tilde{\omega}$$

where the function  $\bar{\omega}(x, y)$  is the solution of boundary value problem (BVP) (5.55), (5.56) with  $u = w$  (we can set  $u = w$  due to above assumptions on the supports of  $u_1$  and  $U$ ). Thus this BVP has the form

$$(5.86) \quad \Delta \bar{\omega} = 0, \quad \bar{\omega}(x, h, t) = 0, \quad \bar{\omega}(x, 0, t) = w_x.$$

This boundary value problem (BVP) defines a linear map  $w \rightarrow \bar{\omega}(w)$ .

Removing the nonlinear terms and ones of order  $\gamma$  and  $\gamma^2$  in (5.83), (5.84), we obtain the linear operator

$$(5.87) \quad Lv = (\bar{L}_1 v, \bar{L}_2 v)^{tr}, \quad v = (\tilde{\omega}, w)^{tr}$$

where the operators  $\bar{L}_k$  are defined by

$$(5.88) \quad \bar{L}_1 v = \nu \Delta \tilde{\omega} - \bar{\omega}(\bar{L}_2 v),$$

$$(5.89) \quad \bar{L}_2 v = \Delta w + \psi_x U_y.$$

Here  $\psi(\tilde{\omega}, w)$  is defined by (5.64) with  $\omega = \bar{\omega}(w) + \tilde{\omega}$  in the right hand side. Below we use a natural decomposition  $\psi = \bar{\psi} + \tilde{\psi}$ , where  $\bar{\psi}, \tilde{\psi}$  are defined by (5.57), (5.58) and (5.59), (5.60), respectively.

The spectral problem for the operator  $L$

$$(5.90) \quad \lambda \tilde{\omega} = \bar{L}_1 v, \quad \lambda w = \bar{L}_2 v$$

can be represented in the standard form. Indeed, the equation for the first component  $v_1 = \tilde{\omega}$  takes the form

$$\lambda \tilde{\omega} = \nu \Delta \tilde{\omega} - \bar{\omega}(\bar{L}_2 v),$$

and, since  $\Delta \bar{\omega} = 0$ ,  $\bar{\omega}(\bar{L}_2 v) = \lambda \bar{\omega}(w)$ , spectral problem (5.90) becomes

$$(5.91) \quad \lambda \omega = \nu \Delta \omega,$$

$$(5.92) \quad \lambda w = \Delta w + \psi_x U_y,$$

where the functions  $w$  and  $\omega$  satisfy the boundary conditions

$$(5.93) \quad w_y(x, y)|_{y=0, h} = 0, \quad \omega(x, h) = 0, \quad \omega(x, 0) = w_x.$$

This spectral problem is investigated in coming sections but first we consider some general properties of  $L$ .

**5.6.  $L$  is a sectorial operator.** In order to apply the standard technique [13], first let us show that the operator  $-L$  is sectorial. We use the following known result [13]: if  $L^{(0)}$  is a self adjoint operator in a Banach space  $X$ ,  $L^{(0)} : X \rightarrow X$  and  $B$  is a linear operator,  $B : X \rightarrow X$  such that  $Dom L^{(0)} \subset Dom B$  and for all  $\rho \in Dom L^{(0)}$

$$(5.94) \quad \|B\rho\| \leq \sigma \|L^{(0)}\rho\| + K(\sigma)\|\rho\|$$

for  $0 < \sigma < 1$  and some constant  $K(\sigma)$ , then  $L^{(0)} + B$  also is a sectorial operator.

Let us define the unperturbed operator  $L^{(0)}$  by the relations

$$\bar{L}_1^{(0)}(\omega, w)^{tr} = \nu \Delta \omega, \quad \bar{L}_2^{(0)}(\omega, w)^{tr} = \Delta w$$

with domains defined by (5.50) and (5.48), respectively. The operator  $B$  is given then by

$$B(\omega, w)^{tr} = (-\bar{\omega}(\bar{L}_2 v), \psi_x U_y)^{tr}.$$

The spectral problem for  $L^{(0)}$  is given by (5.91), (5.92) and (5.93) with  $U = 0$ . The corresponding eigenfunctions are

$$w_{k,m}(x, y) = \cos(m\pi y h^{-1}) \exp(ikx), \quad \omega_m = 0, \quad k, m = 0, 1, \dots,$$

with the eigenvalues  $\lambda_{k,m} = -m^2\pi^2 h^{-2} - k^2$ , and

$$w_{k,n}(x, y) = 0, \quad \omega_{k,n} = \sin(n\pi y h^{-1}) \exp(ikx), \quad n = 1, 2, \dots, \quad k = 0, 1, \dots,$$

with the eigenvalues  $\bar{\lambda}_{k,n} = -\nu n^2\pi^2 h^{-2} - k^2$ .

LEMMA 5.4. *Under condition (5.80)  $L$  is a sectorial operator.*

The operator  $L^{(0)}$  is self-adjoint, the spectrum is discrete and lies in the interval  $(-\infty, 0)$ . Therefore,  $L^{(0)}$  is a sectorial.

Let us check estimate (5.94). First we estimate  $\bar{\omega}(\bar{L}_2 v)$ . Since  $U$  satisfies (5.80), one has  $\bar{\omega}(\bar{L}_2 v) = \bar{\omega}(\Delta w)$ . Thus, for  $\alpha \in (1/2, 1)$  and for any  $\sigma > 0$  by Lemma 5.1 we obtain

$$(5.95) \quad \|\bar{\omega}(\Delta w)\| \leq c \|\Delta w\|_\alpha \leq c \|w\|_{1+\alpha} \leq K(\sigma) \|w\|_1 + \sigma \|\Delta w\|_1.$$

Let us consider the second component  $\psi_x U_y$  of  $B$ , where  $\psi_x = \bar{\psi}_x + \tilde{\psi}_x U_y$ . Since  $U(y)$  is a smooth function, and  $\tilde{\psi}$  is a solution of boundary value problem (5.59), (5.60) one obtains

$$(5.96) \quad \|\tilde{\psi}_x U_y\|_1 \leq c \|\tilde{\omega}_x\| \leq \sigma \|\Delta \tilde{\omega}\| + K_1(\sigma) \|\tilde{\omega}\|$$

for some  $K_1(\sigma)$ . The function  $\bar{\psi}$  is a solution of BVP (5.57), (5.58). Therefore, we can represent  $\bar{\psi}$  as

$$\bar{\psi}(x, y) = \int_0^{2\pi} \Gamma(x-s, y) w_s(s, 0) ds,$$

where  $\Gamma$  is analytic in  $x-s, y$  for  $y > \delta_1 > 0$ . Since  $U$  satisfies (5.80) this relation entails

$$\|\bar{\psi}_x U_y\|_1 \leq C_2 \int_0^{2\pi} |w(s, 0)|^2 ds.$$

By estimate (5.53) for traces one has

$$(5.97) \quad \|\bar{\psi}_x U_y\|_1 \leq C_3 \|w\|_{1/2} \leq \sigma \|w\|_1 + K_2(\sigma) \|w\|.$$

This estimate together with (5.96) and (5.95), proves the lemma.

**5.7. Existence and uniqueness.** The fact that the linear operator  $L$  is sectorial and the  $C^1$  smoothness of the maps  $F$  and  $Z$  entail that system (5.68) -(5.69) defines a local  $C^1$ -smooth semiflow. For results on global existence see [24]. We do not use these results in this paper. The global existence on time interval  $[0, +\infty)$  will be proved only for trajectories, which lie in a small open neighborhood of the positively invariant manifold  $\mathcal{M}_n$  (see definition 3.1).

**5.8. Resolvent of  $L$  is a compact operator.** Let us prove that the resolvent of  $L$  is a compact operator.

LEMMA 5.5. *For sufficiently large  $\nu > 0$  the resolvent  $(L - \lambda)^{-1}$  is a compact operator from  $\mathcal{H}$  to  $\mathcal{H}$  for some  $\lambda$ .*

**Proof.** Let us take  $\lambda = \nu^2$ . Consider the boundary value problem that defines the resolvent:

$$(5.98) \quad \lambda\omega - \nu\Delta\omega = f, \quad \omega(x, h) = 0, \quad \omega(x, 0) = w_x(x, 0),$$

$$(5.99) \quad \lambda w - \Delta w = \psi_x U_y + g, \quad w_y(x, y)|_{y=0, h} = 0,$$

where

$$(5.100) \quad \Delta\psi = -\omega, \quad \psi(x, 0) = 0, \quad \psi(x, h) = 0$$

and  $f \in H$ ,  $g \in H_1$ , i.e.,

$$(5.101) \quad \|f\| + \|g\|_1 < C_0.$$

We assume that  $\langle g, 1 \rangle = 0$ . We represent  $\omega$  as a sum  $\omega = \bar{\omega} + \tilde{\omega}$ , where

$$(5.102) \quad \lambda\tilde{\omega} - \nu\Delta\tilde{\omega} = f, \quad \tilde{\omega}(x, h) = 0, \quad \tilde{\omega}(x, 0) = 0,$$

$$(5.103) \quad \lambda\bar{\omega} - \nu\Delta\bar{\omega} = 0, \quad \bar{\omega}(x, h) = 0, \quad \bar{\omega}(x, 0) = w_x(x, 0).$$

In order to prove the lemma, it is sufficient to obtain for some  $\alpha > 0$  the following estimates:

$$(5.104) \quad \|\tilde{\omega}\|_\alpha \leq c\|f\|,$$

$$(5.105) \quad \|w\|_{1+\alpha} \leq c_2(\|f\| + \|g\|_1),$$

$$(5.106) \quad \|\bar{\omega}\|_\alpha \leq c_1(\|f\| + \|g\|_1).$$

Since  $\lambda = \nu^2$  the solution  $\tilde{\omega}$  of eq. (5.102) satisfies estimate (5.104). Note that  $\tilde{\omega}$  does not depend on  $w$ . We use the natural decomposition  $\psi = \tilde{\psi} + \bar{\psi}$  defined by (5.57), (5.58), (5.59) and (5.60). Then BVP (5.99) can be rewritten in an operator form as

$$(5.107) \quad w = A_\nu w + \tilde{g},$$

where

$$A_\nu w = (\nu^2 - \Delta)^{-1}(\tilde{\psi}_x(w(\cdot, \cdot))U_y)$$

defines a linear operator  $A_\nu : H_1 \rightarrow H_1$  and

$$\tilde{g} = (\nu^2 - \Delta)^{-1}(\tilde{\psi}_x U_y + g).$$

Let us prove that  $A_\nu$  is a contraction in the space  $H_1$  for sufficiently large  $\nu$ . We use estimate (5.97) that gives

$$(5.108) \quad \|A_\nu w\|_1 \leq \nu^{-1}(\sigma\|w\|_1 + K\|w\|)$$

for some  $K > 0$ , which does not depend on  $\nu$ . This estimate shows that the operator  $A_\nu$  is a contraction in  $H_1$  for sufficiently large  $\nu$ . Therefore, the solution of (5.107) exists and satisfies

$$(5.109) \quad \|w\|_{1+\alpha} \leq C\|\tilde{g}\|_\alpha.$$

One has

$$(5.110) \quad \|\tilde{g}\|_\alpha \leq c(\|\tilde{\psi}_x U_y\|_\alpha + \|g\|_1).$$

Note that  $U_y$  is a smooth bounded function and the solution  $\tilde{\psi}$  of (5.102) satisfies  $\|\tilde{\psi}_x\|_\alpha \leq c_2\|f\|$  for some  $c_2 > 0$ . Therefore,

$$(5.111) \quad \|\tilde{g}\|_\alpha \leq c_3(\|f\| + \|g\|_1).$$



Estimates (5.109), (5.110) and (5.111) prove (5.105). Estimate (5.106) follows from (5.105) by the embedding for traces. The lemma is proved.

This lemma implies, according to the well known result (see [14], Ch. III, Theorem 6.29), that the spectrum of  $L$  is discrete (it consists of isolated eigenvalues), each eigenvalue has a finite multiplicity  $n(\lambda)$ , and the resolvent  $\mathcal{R}(\lambda)$  is a compact operator for all  $\lambda$ , where  $\mathcal{R}(\lambda)$  is defined. We investigate the spectrum in the next section.

## 6. Spectrum of the main linear operator

**6.1. Some preliminaries.** Let us consider spectral problem (5.91), (5.92) and (5.93). For any  $U(y)$  this problem has the trivial eigenfunction  $e_0 = (0, 1)^{tr}$ , where  $\omega = 0, w = 1$ , with the zero eigenvalue  $\lambda$ . We consider eigenfunctions  $e(x, y, \lambda)$  with eigenvalues  $\lambda \in \mathbf{C}_{1/2}$ , where  $\mathbf{C}_a$  denotes the half-plane

$$(6.112) \quad \mathbf{C}_a = \{\lambda \in \mathbf{C} : \operatorname{Re} \lambda > -a\}.$$

Since  $U$  depends only on  $y$ , we seek the eigenfunctions of the form

$$(6.113) \quad w(x, y, \lambda) = w_k(y, \lambda) \exp(ikx),$$

$$(6.114) \quad \psi(x, y, \lambda) = \psi_k(y, \lambda) \exp(ikx), \quad \omega(x, y, \lambda) = \omega_k(y, \lambda) \exp(ikx).$$

For  $\omega_k, \psi_k$  and  $w_k$  one obtains the following boundary value problem:

$$(6.115) \quad \frac{\partial^2 \omega_k}{\partial y^2} - k_\nu^2 \omega_k = 0, \quad \omega_k(h, \lambda) = 0, \quad \omega_k(0, \lambda) = ikw_k(0, \lambda),$$

where  $k_\nu^2 = k^2 + \lambda/\nu$ ,

$$(6.116) \quad \frac{\partial^2 \psi_k}{\partial y^2} - k^2 \psi_k = -\omega_k, \quad \psi_k(h, \lambda) = 0, \quad \psi_k(0, \lambda) = 0,$$

$$(6.117) \quad \frac{\partial^2 w_k}{\partial y^2} - \bar{k}^2 w_k = ikU_y(y)\psi_k, \quad \frac{\partial w_k(y, \lambda)}{\partial y} \Big|_{y=0, h} = 0,$$

where  $\bar{k} = \sqrt{k^2 + \lambda}$ . Let us suppose, without loss of generality, that  $k > 0$ , and  $\operatorname{Re} \bar{k} > 0$  for  $\lambda \in \mathbf{C}_{1/2}$ , since  $w_{-k}$  are functions, complex conjugate to  $w_k$  and  $\bar{k}$  is involved only via  $\bar{k}^2$ . We assume that

$$(6.118) \quad h = 10 \log \nu, \quad \nu \gg 1.$$

The solution of problem (6.115), (6.116) is defined by

$$(6.119) \quad \omega_k(y, \lambda) = \beta_k \frac{\sinh(\bar{k}_\nu(h-y))}{\sinh(\bar{k}_\nu h)}, \quad \beta_k(\lambda) = ikw_k(0, \lambda),$$

$$(6.120) \quad \psi_k(y, \lambda) = -\nu \beta_k \lambda^{-1} \Phi_k(y, \lambda),$$

$$(6.121) \quad \Phi_k(y, \lambda) = \frac{\sinh(kh) \sinh(\bar{k}_\nu(h-y)) - \sinh(\bar{k}_\nu h) \sinh(k(h-y))}{\sinh(\bar{k}_\nu h) \sinh(kh)}.$$

Note that relation (6.120) is correctly defined for all  $\lambda \in \mathbf{C}_{1/2}$ , in particular, for  $\lambda = 0$ . Indeed, , for small  $\lambda$

$$(6.122) \quad \bar{k}_\nu - k = \sqrt{k^2 + \lambda \nu^{-1}} - k = \lambda(2\nu k)^{-1} + O(\lambda^2 \nu^{-2} k^{-3})$$

that gives

$$(6.123) \quad \psi_k(y, \lambda) = \beta_k(\lambda) \frac{y \sinh(kh) \cosh(k(h-y)) - h \sinh(ky)}{2k \sinh^2(kh)} + \phi(y, k),$$

where

$$|\phi(y, k, \lambda)| < c|w_k(0, \lambda)||\lambda|(k\nu)^{-1}, \quad 0 < y < h.$$

For large  $\nu_0$  and  $|\lambda| \ll \nu$  assumptions (6.118) allow us to simplify (6.119) and (6.120). By (6.119) we obtain then

$$(6.124) \quad \omega_k(y, \lambda) = \beta_k(\lambda)(\exp(-ky) + \tilde{\omega}_k(y, \nu)),$$

where for each  $s \in (0, 1)$  and  $|\lambda| < \nu^s$

$$(6.125) \quad |\tilde{\omega}_k(y, \nu)| < C_s(|\lambda|(k\nu)^{-1} \exp(-ky) + \exp(-kh)), \quad y \in [0, h],$$

where  $C_s > 0$  are constants independent of  $s$  and  $k$ . This estimate and (6.123) give

$$(6.126) \quad \psi_k(y, \lambda) = \beta_k(\lambda)(\bar{\psi}_k(y, \lambda) + \tilde{\xi}_k(y, \lambda)),$$

where

$$(6.127) \quad \bar{\psi}_k(y, \lambda) = \frac{y}{2k} \exp(-ky),$$

and for  $|\lambda| < \nu^s$

$$(6.128) \quad |\tilde{\xi}_k(y, \lambda)| < \bar{C}_s(|\lambda|k^{-1}\nu^{-1} \exp(-ky) + \exp(-kh)), \quad y \in (0, h),$$

where constants  $\bar{C}_s > 0$  are uniform in  $k, \nu$ .

To investigate (6.117), we apply a lemma.

LEMMA 6.1. *Let us consider the boundary value problem on  $[0, h]$  defined by*

$$(6.129) \quad w_{yy} - \bar{k}^2 w = f(y), \quad y \in [0, h],$$

$$(6.130) \quad w_y(y)|_{y=0, h} = 0.$$

Then

$$(6.131) \quad w(0) = - \int_0^h f(y) \rho_{\bar{k}}(y) dy,$$

where

$$(6.132) \quad \rho_{\bar{k}}(y) = \frac{\cosh(\bar{k}(h-y))}{\bar{k} \sinh \bar{k} h}.$$

To prove it, we multiply both the right hand and the left hand sides of eq. (6.129) by  $\rho_{\bar{k}}$  and integrate by parts in the left hand side. Note that

$$(6.133) \quad |\rho_{\bar{k}}(y) - \bar{\rho}_{\bar{k}}(y)| < \bar{k}^{-1} \exp(-\bar{k}h), \quad \bar{\rho}_{\bar{k}}(y) = \bar{k}^{-1} \exp(-\bar{k}y).$$

**6.2. Main result on spectrum of operator  $L$ .** Let us formulate the assertion.

**PROPOSITION 6.2.** *Let assumptions (6.118) hold,  $N$  be a positive integer and  $K_N = \{1, \dots, N\} \subset \mathbf{Z}_+$ . Then there exists a  $C^\infty$  smooth function  $U(y) = U_N(y, \nu)$  satisfying (5.80) and such that for sufficiently large  $\nu > \nu_0(N) > 0$  the eigenfunctions  $\lambda(k, \nu)$  of BVP (6.115), (6.116) and (6.117) satisfy*

(i)

$$(6.134) \quad \lambda(k, \nu) = 0 \quad k \in K_N,$$

(ii)

$$(6.135) \quad \operatorname{Re} \lambda(k, \nu) < -\delta_N \quad k \notin K_N,$$

where positive  $\delta_N$  is uniform in  $\nu$ .

**Proof.** We use Lemma 6.1 to obtain a nonlinear equation for the eigenvalues  $\lambda(k)$  of the boundary value problem (6.115)-(6.117). As a result, one has

$$(6.136) \quad -k^2 \int_0^h \psi_k(y, \lambda) \rho_{\bar{k}}(y) U_y(y, \nu) dy = \beta_k(\lambda).$$

Note that the operator  $L$  is not self-adjoint. Therefore, there are possible complex eigenvalues  $\lambda$ , i.e., complex roots of (6.136). Moreover, let us note that, according to (6.120), if  $\beta_k(\lambda) = 0$ , then eq. (6.136) is satisfied. In this case (6.115) entails that

$$(6.137) \quad \frac{d^2 \omega_k(y)}{dy^2} - k_\nu^2 \omega_k(y) = 0, \quad \omega_k(h) = 0, \quad \omega_k(0) = 0,$$

therefore  $k_\nu^2 = -(n\pi/h)^2$ , where  $n$  is an integer. This gives  $\lambda = -\nu((n\pi/h)^2 + k^2) < -1/2$ . These eigenvalues  $\lambda$  correspond to trivial solutions of the eigenfunction problem with  $\lambda \notin \mathbf{C}_{1/2}$ . Therefore, without loss of generality we can set  $\beta_k(\lambda) = 1$  in eq. (6.136).

The plan of the proof is as follows. We consider the two cases: **(I)**  $|\lambda| < \nu^{3/4}$  and **(II)**  $|\lambda| > \nu^{3/4}$ . In the first case we can simplify equation (6.136), in the second case a rough estimate shows that eq. (6.136) has no solutions.

Let us start with the case **I**. To simplify our statement, we first consider a formal limit of equation (6.136) as  $\nu \rightarrow +\infty$ . Using (6.127) and (6.133) one obtains that this limit has the form

$$(6.138) \quad \int_0^{+\infty} y \bar{U}_y(y) \exp(-(k + \bar{k})y) dy = 2\bar{k}/k,$$

where  $\bar{U} = \lim U_{K_N}(y, \nu)$  as  $\nu \rightarrow +\infty$ . We set  $\bar{U} = V(y, d)$ , where  $V$  is a function of a special form that depends on some parameters  $d = (d_1, \dots, d_N)$ . Consider  $C^\infty$ -mollifiers  $\delta_\epsilon(y)$  such that  $\delta_\epsilon \geq 0$ , the support  $\operatorname{supp} \delta_\epsilon(y)$  is  $(-\epsilon, \epsilon)$  and

$$(6.139) \quad \int_{-\epsilon}^{\epsilon} \delta_\epsilon(y) dy = 1,$$

$$(6.140) \quad \sup |D_y^k \delta_\epsilon(y)| < c_k \epsilon^{-(k+1)}, \quad k = 0, 1, 2.$$

Let us define the function  $V(y, d)$  on  $(0, \infty)$  by

$$(6.141) \quad V_y(y, d) = 2y^{-1}(\delta_\kappa(y - z_0) + \mu\chi(y - z_0)yW_N(y, d)), \quad V(0, d) = 0,$$

where  $\chi(z)$  is the step function such that  $\chi(z) = 1$   $z > 0$  and  $\chi(z) = 0$   $z \leq 0$ ,  $W_N$  is a polynomial in  $y$  of the degree  $N + 1$  with coefficients depending on some parameters  $d = (d_1, d_2, \dots, d_N)$ , and

$$(6.142) \quad \mu = \kappa^{2/3}, \quad z_0 = 5\kappa,$$

where  $\kappa$  is a small parameter independent of  $\nu$  as  $\nu \rightarrow +\infty$ . Coefficients of the polynomial  $W_N$  and  $d_j$  are assumed to be bounded:

$$(6.143) \quad W_N(y, d) = \sum_{j=0}^N b_j(d) y^j, \quad |b_j(d)| < C_j, \quad |d_j| < 1/2.$$

We set  $U(y, \kappa) = \int_0^y V_s ds$  then  $U$  satisfies condition (5.80).

To investigate (6.138) it is useful to introduce the variable

$$(6.144) \quad p = k + \bar{k} = k + (k^2 + \lambda)^{1/2}.$$

Then eq. (6.138) can be rewritten as

$$(6.145) \quad \frac{p}{k} = 2 + S(p, k, d),$$

where

$$(6.146) \quad S(p, k, d) = \mu G(p, d) + g_\kappa(p) + \tilde{g}_\kappa(p),$$

$$(6.147) \quad g_\kappa(p) = -1 + \int_0^\infty \delta_\kappa(y - z_0) \exp(-py) dy,$$

$$(6.148) \quad \tilde{g}_\kappa(p) = -\mu \int_0^{z_0} y W_N(y, d) \exp(-py) dy,$$

and

$$(6.149) \quad G(p, d) = \int_0^\infty y W_N(y, d) \exp(-py) dy.$$

We suppose that  $\bar{k} > 0$  thus  $\operatorname{Re} p > k$ . Therefore, we can investigate (6.145) in the domain

$$(6.150) \quad \mathbf{C}_{1/2, k} = \{p \in \mathbf{C} : p = \sqrt{k^2 + \lambda} + k, \operatorname{Re} \lambda > -1/2, \operatorname{Re} p > k\}.$$

Note that

$$(6.151) \quad \operatorname{Re} p = k + \sqrt{k^2 + \operatorname{Re} \lambda + (\operatorname{Im} p)^2},$$

this shows that in  $\mathbf{C}_{1/2, k}$  we have  $\operatorname{Re} p > 2k - 1/2$ . We can choose a polynomial  $W_N$  such that

$$(6.152) \quad G(p, d) = p^{-2} (-1)^{N+1} \prod_{j=1}^N \left( \frac{1}{p} - \frac{1}{2j + d_j} \right).$$

Let us formulate an auxiliary assertion.

LEMMA 6.3. *One has*

$$(6.153) \quad \operatorname{Re} g_\kappa(p) \leq -\min\{4 \exp(-4) \kappa \operatorname{Re} p, \quad 1/2\}.$$

**Proof.** Estimate (6.153) follows from (6.142) and (6.147). Indeed, due to (6.139) we have

$$\operatorname{Re} g_\kappa(p) = \int_0^h \delta_\kappa(y - z_0)(\operatorname{Re} \exp(-py) - 1)dy \leq \int_0^h \delta_\kappa(\exp(-\operatorname{Re} p y) - 1)dy$$

that according to (6.142) gives  $\operatorname{Re} g_\kappa(p) < \exp(-4\kappa \operatorname{Re} p) - 1 := J(\operatorname{Re} p)$ . Moreover,  $|J(\operatorname{Re} p)| \leq 4 \exp(-4) \operatorname{Re} p |\kappa|$  for  $0 < \operatorname{Re} p < \kappa^{-1}$  and  $J(\operatorname{Re} p) < 1/2$  for  $|\operatorname{Re} p| \geq \kappa^{-1}$ .

Let us show that in the case  $|p| > \kappa^{-3/4}$  equation (6.145) has no solutions with  $\operatorname{Re} \lambda > -c_0 \kappa$ .

LEMMA 6.4. *If  $|p| > \kappa^{-3/4}$  then for sufficiently small  $\kappa$  solutions of (6.145) satisfy*

$$(6.154) \quad \operatorname{Re} p < 2k - c_1 k^2 \kappa^{1/4}$$

For the corresponding  $\lambda_k$  one has

$$(6.155) \quad \operatorname{Re} \lambda_k < -c_2 k^2 \kappa^{1/4}$$

**Proof.** Relation (6.151) shows that  $\operatorname{Re} p > \operatorname{Im} p$ , thus  $|p| > \kappa^{-3/4}$  entails  $4 \operatorname{Re} p > \kappa^{-3/4}$ . Then estimate (6.153) implies  $\operatorname{Re} g_\kappa(p) < -c_2 \kappa^{1/4}$ . Moreover,  $\mu \operatorname{Re} G(p, d) < -c_3 \mu$  and for  $\operatorname{Re} p > -1/2$

$$(6.156) \quad \tilde{g}_\kappa(p) < c_4 \mu |z_0| < c_5 \kappa^{5/3}.$$

These estimates entail that  $\operatorname{Re} S(p, k, d) < -c_4 \kappa^{1/4}$  and, therefore, (6.154) holds that, in turn, by (6.151) gives us (6.155).

Consider the case  $|p| < \kappa^{-3/4}$ . Let us introduce a new unknown  $\tilde{p}$  by  $(2 + \tilde{p})k = p$ . Then equation (6.145) can be rewritten as

$$(6.157) \quad \tilde{p} = H(\tilde{p}, k),$$

where

$$(6.158) \quad H(\tilde{p}, k) = (\bar{g}_\kappa((2 + \tilde{p})k) + \mu G((2 + \tilde{p})k, d))$$

and  $\bar{g}_\kappa = g_\kappa + g_\kappa$ . Let us prove an estimate of solutions to (6.157).

LEMMA 6.5. *In the domain  $\mathcal{D}_{\kappa, k} = \{p : p \in \mathbf{C}_{1/2, k}, |p| < c_1 \kappa^{-3/4}\}$  solutions of (6.157) satisfy*

$$(6.159) \quad |\tilde{p}| < C_1 \kappa^{1/4}.$$

**Proof.** To prove this lemma, we note that if  $p \in \mathcal{D}_{\kappa, k}$  then

$$(6.160) \quad |\bar{g}_\kappa(p)| < C_2 \kappa^{1/4}, \quad |G(p, d)| < C_3.$$

Moreover, estimate (6.156) holds. Therefore, one obtains

$$(6.161) \quad |H(\tilde{p}, k)| < C_5(\mu + \kappa^{1/4}).$$

Now (6.158) and (6.157) show that  $\tilde{p}$  satisfies (6.159). The lemma is proved.

Let us consider equation (6.157). Using the last lemma we note that, to resolve this equation, we can apply a simple perturbation theory. Relations (6.148), (6.147), (6.146) and (6.152) show that in the domain  $\mathcal{D}_{\kappa, k}$  one has

$$\left| \frac{\partial H(\tilde{p}, j)}{\partial \tilde{p}} \right| < C_6 \mu.$$

Now Lemma 6.5 and the implicit function theorem entail that for sufficiently small  $\kappa$  all roots  $\tilde{p}$  of eq. (6.157) lie in  $\mathcal{D}_{\kappa,k}$  and can be found by contracting mappings. For each fixed  $k$  the solution  $\tilde{p}_k$  of eq. (6.157) is unique in  $\mathcal{D}_{\kappa,k}$ .

LEMMA 6.6. *For sufficiently small  $\kappa$  we can choose  $d_j \in (-1/2, 1/2)$ ,  $j = 1, \dots, N$ , such that for each  $k \in \{1, \dots, N\}$  eq. (6.157) has a unique solution  $\tilde{p}_k = 0$  and for  $k > N$  any solution of eq. (6.157) satisfy*

$$(6.162) \quad \operatorname{Re} \tilde{p}_k < -c\kappa^s, \quad s > 0.$$

**Proof.** Due to Lemma 6.4, we can assume  $|p| < \kappa^{-3/4}$  and then, according to Lemma 6.5 for all  $k$  solutions  $\tilde{p}_k$  lie in the domain defined by inequality (6.159).

Assume that  $k \in \{1, \dots, N\}$ . Let us fix  $k$  and consider  $p \in \mathcal{D}_{\kappa,k}$ , i.e.,  $p$  close to  $2k$ . Then for  $\tilde{p}$  satisfying (6.159) from (6.152) we obtain the following asymptotic for  $G(p, d)$ :

$$G(p, d) = \mu(-1)^{N+1}(\bar{a}_k + \tilde{a}_k(\tilde{p}_k, d))(d_k - k\tilde{p}_k),$$

where

$$\bar{a}_k = \prod_{j=1, j \neq k}^{N+1} \left( \frac{1}{2k} - \frac{1}{2j} \right),$$

and  $\tilde{a}_k(\tilde{p}_k, d)$  is an analytic function such that  $|\tilde{a}_k| = O(|\tilde{p}_k| + |d|)$  for small  $\tilde{p}_k, d$ . Eq. (6.157 takes the form

$$(6.163) \quad \tilde{p}_k = R_k(\tilde{p}_k, d, \kappa) + \mu(-1)^{M+1}\bar{a}_k d_k,$$

where  $R_k$  is an analytic function of  $\tilde{p}_k$  and  $d$  for small  $|\tilde{p}_k|, |d|$  such that

$$\sup_{\tilde{p}_k, d: |d| < \mu^{1/2}, \tilde{p}_k \in \mathcal{D}_{\kappa,k}} (|R_k| + |\operatorname{grad}(R_k)|) < c\kappa^s + c_2|b - b_c|,$$

for some  $s > 0$ . Therefore, for small  $b - b_c$  we can apply the Implicit Function Theorem to find need  $d_k$  such that the root  $\tilde{p}_k$  of (6.163) equal zero.

Let us consider the case  $k > N$ . We observe that for  $k > N$  and  $\tilde{p}$  satisfying (6.159) we have  $\mu \operatorname{Re} G < -c_2\kappa^{2/3}$ ,  $\operatorname{Re} \tilde{g}_\kappa(p) < -c\kappa$ , and thus  $\operatorname{Re} \tilde{p} < -c_2k\kappa^{2/3}$ . The lemma is proved.

Finally, we have obtained need estimates of solutions (6.136) for the case **I**,  $\beta_k = 1$  and  $\nu = +\infty$ . To finish our investigation of equation (6.136) for the case of large  $\nu$ , we compare equations (6.136) and its formal limit (6.138). We assume that the function  $U = V(y, d)$  is defined as above, by (6.141).

We observe that for  $\beta_k = 1$  eq. (6.136) can be rewritten as

$$(6.164) \quad k\bar{k}^{-1}(\lambda) \int_0^{+\infty} yV(y, d) \exp(-(k + \bar{k}(\lambda))y) dy = 2 - R_k(\lambda, \nu, d),$$

where  $R_k = I_k + J_k$  and

$$(6.165) \quad I_k = k^2 \int_h^{+\infty} \bar{\psi}_k(y, \lambda) \bar{\rho}_{\bar{k}} V_y(y, d) dy,$$

$$(6.166) \quad J_k = -k^2 \int_0^h (\psi_k(y, \lambda) \rho_{\bar{k}} - \bar{\psi}_k(y) \bar{\rho}_{\bar{k}}) V_y(y, d) dy.$$

Under assumption (6.118),  $\lambda \ll \nu^s$  for  $s \in (0, 1)$  and for sufficiently large  $\nu$  the term  $I_k$  satisfies the estimate

$$(6.167) \quad |I_k| < c_1 k^2 h^{N+3} \exp(-kh) < c_2 \nu^{-4},$$

which is uniform in  $k$ . To estimate  $J_k$  we use the inequality

$$(6.168) \quad |J_k| \leq k^2 \int_0^h (|(\psi_k(y, \lambda) - \bar{\psi}_k(y))\rho_{\bar{k}}| + |\bar{\psi}_k(y)(\bar{\rho}_{\bar{k}}(y) - \rho_k(y))|) |V_y(y, d)| dy.$$

Now we apply (6.126), definition (6.141) of  $V_y$ , estimates (6.127), (6.128) and (6.133) that gives

$$\sup_{y \in [z_0, h]} |V_y(y, d)| < c_3(\kappa^{-2} + h^{N+3}) \exp(-k\kappa).$$

Then we see that

$$\begin{aligned} |J_k| &< c_4 k^2 (\kappa^{-2} + h^{N+3}) (k^2 \bar{k}^{-1} \exp(-\bar{k}h) + k \bar{k}^{-1} \lambda \nu^{-1}) \exp(-k\kappa) < \\ &< c_5 (1 + \kappa^{-2}) \nu^{-1/2} \end{aligned}$$

for some  $c_5 > 0$  and sufficiently large  $\nu > \nu_0(\kappa)$ . Note that in this estimate the constant  $c_5$  is uniform in  $k$ . We obtain finally the uniform in  $k$  estimate

$$(6.169) \quad |R_k(\lambda, \nu, d)| < c_6 \nu^{-1/2}.$$

Therefore, the analysis of equation (6.164) can be made by the same arguments as above in the case of formal limit  $\nu \rightarrow +\infty$  that allows us to prove the lemma.

**LEMMA 6.7.** *Let assumptions (6.118) hold,  $N$  be a positive integer and  $V(y, d)$  is defined by (6.141). Moreover, let  $|\lambda| < \nu^{3/4}$ . Then we can choose such parameters  $\kappa$  and  $d$  in relation (6.141) that for sufficiently large  $\nu$  the roots  $\lambda(k, \nu)$  of equation (6.164), which lie in the domain  $|\lambda| < \nu^{3/4}$ , satisfy*

$$(6.170) \quad \lambda(k, \nu) = 0 \quad k = 1, \dots, N$$

and

$$(6.171) \quad \operatorname{Re} \lambda(k, \nu) < -\delta_N \quad k > N,$$

where a positive  $\delta_N$  is uniform in  $k$  and in  $\nu$  as  $\nu \rightarrow \infty$ .

**Proof.** We repeat the proof of Lemma 6.6. Since  $R_k(\lambda, \nu, d)$  satisfies estimate (6.6), we obtain new  $d_j = \bar{d}_j(\nu)$ , which are small perturbations of  $d_j$  obtained in Lemma 6.6. We have  $\bar{d}_j(\nu) - d_j = \tilde{d}_j(\nu)$ , where  $\tilde{d}_j(\nu) \rightarrow 0$  as  $\nu \rightarrow +\infty$ . Therefore, the  $\sup |U_y(y, \nu) - V_y(y, d)| \rightarrow 0$  as  $\nu \rightarrow +\infty$  and (6.170) holds. Inequalities (6.170) are fulfilled for sufficiently large  $\nu$ . It follows from equation (6.164) and estimate (6.6), which is uniform in  $k$ . The lemma is proved.

**Case II.**

Let us consider now the second case **II**. Relations (6.120) and (6.133) imply that

$$|\psi_k(y, \lambda)| < c_1 \nu / |\lambda|, \quad |\rho_{\bar{k}}(y)| < c_2 \exp(-\bar{k}y) |\bar{k}|^{-1},$$

where  $|\bar{k}| > |\lambda|^{1/2}$ . Moreover,  $|yU_y| < Ch^{M+2}$ . Thus the left hand side of eq. (6.136) is not more than  $R = C\nu \log(\nu)^{M+2} \bar{k} |\bar{k}|^{-1} |\lambda|^{-3/2}$ . For  $\nu \rightarrow +\infty$  and  $|\lambda| > \nu^{3/4}$  one has  $R < c\nu^{-1/8}$ . Therefore, eq. (6.136) has no solutions in the case **II**.

The assertion of Proposition 6.2 follows from this lemma and Lemma 6.7.

**6.3. Eigenfunctions of  $L$  with zero eigenvalues.** Let us consider the eigenfunctions  $e_k$  of  $L$  with the zero eigenvalues. We have  $2N+1$  eigenfunctions including the trivial one  $e_0 = (0, 1)$ . All the rest eigenfunctions have the form

$$(6.172) \quad e_k = \exp(ikx)(\omega_k, \theta_k)^{tr}, \quad e_{-k} = \exp(-ikx)(\omega_{-k}, \theta_{-k})^{tr},$$

where  $k = 1, 2, \dots, N$  and

$$(6.173) \quad \omega_k = ikA_k \frac{\sinh(k(h-y))}{\sinh(kh)}, \quad i = \sqrt{-1},$$

$$(6.174) \quad \theta_k = A_k \Theta_k(y),$$

where  $A_k(\nu)$  are constants. The functions  $\Theta_k$  are defined by

$$(6.175) \quad \frac{d^2 \Theta_k(y)}{dy^2} - k^2 \Theta_k = ik \Psi_k(y) U_y(y),$$

where  $\Psi_k$  is given by (6.123) with  $\lambda = 0$  and  $w_k(0, \lambda) = 1$ . Thus,  $\Psi_k$  is defined

$$(6.176) \quad \Psi_k(y) = i \frac{y \sinh(kh) \cosh(k(h-y)) - h \sinh(ky)}{2 \sinh^2(kh)}.$$

Asymptotics of  $\omega_k$  is given by (6.114). Using relations for  $U$ , estimates from the previous subsection and the definition of  $\mu, \kappa, z_0, z_1$ , one has the following asymptotics

$$(6.177) \quad \Theta_k(y) = k^{-1} \exp(-ky)(1 + \tilde{r}_k(y, \nu)), \quad y \in (z_0, h),$$

where

$$(6.178) \quad \sup_{y \in [z_0, h]} |\tilde{r}_k(y, \nu)| < C_1 \nu^{-1/100}.$$

In coming subsection we construct the operator  $L^*$ , formally conjugate to  $L$ , and the corresponding eigenfunctions.

**6.4. Conjugate spectral problem.** Consider spectral problem for conjugate operator  $L^*$ . We follow [30] but use the vortex-stream formulation.

Let us denote by  $\rho$  and  $\tilde{\rho}$  the pairs  $\rho = (\omega, u)^{tr}$  and  $\tilde{\rho} = (z, v)^{tr}$ . Let us calculate the quadratic form  $(L\rho, \tilde{\rho})$ , where  $(\rho, \tilde{\rho})$  denotes a natural inner product defined by

$$(\rho, \tilde{\rho}) = \langle \omega, \tilde{\omega} \rangle + \langle u, \tilde{u} \rangle.$$

We obtain the relations

$$(6.179) \quad \langle \Delta u + \Psi_x U_y, v \rangle = \langle u, \Delta v \rangle - \langle \omega, \Phi(z)_x \rangle + J,$$

where

$$J = \int_0^{2\pi} (u_y(x, y)v(x, y) - u(x, y)v_y(x, y))dx \Big|_{y=0}^{y=h},$$

and  $\Phi$  is a unique solution of the boundary value problem

$$(6.180) \quad \Delta \Phi = U_y v,$$

$$(6.181) \quad \Phi(x, 0) = \Phi(x, h) = 0.$$

Note that

$$(6.182) \quad \langle \nu \Delta \omega, z \rangle = \langle \omega, \nu \Delta z \rangle + \nu \int_0^{2\pi} (\omega_y(x, y)z(x, y) - z_y(x, y)\omega(x, y))dx \Big|_{y=0}^{y=h}.$$



Let us compute the eigenfunctions of the conjugate operator  $L^*$ . Note that these eigenfunctions have been found in [30] in the three-dimensional case for linear profiles  $U(y)$ .

Using boundary conditions (2.18) and (2.19) (2.20), by (6.179) and (6.182) we obtain that the discrete spectrum of the operator  $L^*$  is defined by the following equations

$$(6.183) \quad \lambda z = \nu \Delta z - \Phi(v)_x,$$

$$(6.184) \quad \lambda v = \Delta v,$$

under the boundary conditions

$$(6.185) \quad v_y(x, y)|_{y=h} = 0, \quad v_y(x, y)|_{y=0} = \nu z_{xy}(x, y)|_{y=0},$$

$$(6.186) \quad z(x, 0) = z(x, h) = 0.$$

Now the eigenfunctions of  $L^*$  with the zero eigenvalues can be found. We obtain

$$(6.187) \quad \tilde{e}_k = \exp(ikx)(z_k, \tilde{\theta}_k)^{tr}, \quad k \in \{-N, \dots, N\},$$

where

$$(6.188) \quad \tilde{\theta}_k = \tilde{A}_k \bar{\theta}_k(y), \quad \bar{\theta}_k = \frac{\cosh(k(h-y))}{\sinh(kh)},$$

$$(6.189) \quad z_k = \nu^{-1} \zeta_k(y).$$

Here  $\zeta_k$  are defined as solutions of the boundary value problem

$$\begin{aligned} \frac{d^2 \zeta_k}{dy^2} - k^2 \zeta_k &= -ikU(y)\Phi_k(y) \\ \zeta_k(h) &= 0, \quad \frac{d\zeta_k(y)}{dy}|_{y=0} = i\tilde{A}_k, \end{aligned}$$

where  $\Phi_k(y)$  are defined by

$$\frac{d^2 \Phi_k}{dy^2} - k^2 \Phi_k = \tilde{A}_k U(y) \bar{\theta}_k(y), \quad \Phi_k(0) = \Phi_k(h) = 0.$$

We have the estimate

$$(6.190) \quad \|z_k\|_{C^2(\Omega)} < C_0 \nu^{-1}.$$

The exact form of the functions  $z_k$  is not essential. Only expressions for the functions  $\psi_k$  and  $\theta_k, \tilde{\theta}_k$  and estimate (6.190) are involved in the further statement.

Relations (6.188), (6.177) and (6.190) show that one can adjust the constants  $A_k$  and  $\tilde{A}_k$  such that

$$(6.191) \quad \langle e_l, \tilde{e}_j \rangle = \delta_{lj}.$$

Note that  $\tilde{e}_0 = \tilde{A}_0(0, 1)^{tr}$ .

To obtain real value eigenfunctions, we take real and imaginary parts of these complex eigenfunctions. The real parts of the eigenfunctions, where  $\omega_k, \theta_k$  are proportional to  $\sin(kx)$ ,  $\cos(kx)$  respectively, are enabled by the upper index  $+$ , and the imaginary parts, where  $\omega_k, \theta_k$  are proportional to  $\cos(kx)$ ,  $\sin(kx)$ , are denoted by the upper index  $-$ . The real eigenfunctions of  $L$  have the form

$$(6.192) \quad e_k^+ = (\omega_k(y) \sin(kx), \theta_k(y) \cos(kx))^{tr},$$

$$(6.193) \quad e_k^- = (-\omega_k(y) \cos(kx), \theta_k(y) \sin(kx))^{tr}.$$

Respectively, the real eigenfunctions of  $L^*$  are

$$(6.194) \quad \tilde{e}_k^+ = (\tilde{\omega}_k(y) \sin(kx), \tilde{\theta}_k(y) \cos(kx))^{tr},$$

$$(6.195) \quad \tilde{e}_k^- = (-\tilde{\omega}_k(y) \cos(kx), \tilde{\theta}_k(y) \sin(kx))^{tr}.$$

We have the relations

$$(6.196) \quad \tilde{\theta}_k^+ = a_k \cosh(k(h-y)),$$

where  $a_k$  are coefficients.

The next lemma concludes the investigation of spectral properties of the operator  $L$ .

**LEMMA 6.8.** *For sufficiently large  $\nu$  the eigenvalue 0 of the operator  $L$  has the multiplicity  $N$ . The eigenvalue 0 has no generalized eigenfunctions.*

**Proof.** Let us check that generalized eigenfunctions are absent. Since 0 has a finite multiplicity, we can use the Jordan representation. Assume that there exists a generalized eigenfunction  $e_g$ . Then  $Le_g = b = \sum_{j=1}^N b_l e_l$  for some  $b_l$ ,  $l \in \{1, \dots, N\}$ , where  $b \neq 0$ . Then  $\langle b, \tilde{e}_k \rangle = 0$  for all  $k \in \{1, \dots, N\}$ . Eigenfunctions  $\tilde{e}_k$  and  $e_l$  are biorthogonal according to (6.191). This implies that all coefficients  $b_l = 0$  and the lemma is proved.

**6.5. Estimates for semigroup  $\exp(Lt)$ .** The operator  $L$  is sectorial and, according to Lemma 6.2, satisfies the Spectral Gap Condition. Therefore [13] for some  $\beta(\nu) > 0$  and for all  $v = (\tilde{\omega}, w)^{tr}$  such that

$$\langle v, \tilde{e}_j^\pm \rangle = 0, \quad j = 1, \dots, N,$$

and

$$\langle v, \tilde{e}_0 \rangle = 0, \quad \tilde{e}_0 = (0, 1).$$

one has the following estimates

$$(6.197) \quad \|\exp(Lt)v\| \leq C_0(\nu) \exp(-\beta(\nu)t) \|v\|,$$

$$(6.198) \quad \|\exp(Lt)v\|_{\mathcal{H}_\alpha} \leq C_1(\nu) b_\alpha(t) \exp(-\beta(\nu)t) \|v\|$$

Here

$$b_\alpha(t) = t^{-\alpha}, \quad 0 < t \leq \beta^{-1}(\nu)$$

and

$$b_\alpha(t) = \beta^{-\alpha}, \quad t > \beta^{-1}(\nu).$$

Estimates (6.197) and (6.198) are important in the proof of existence of the invariant manifold.

### 7. Finite dimensional invariant manifold

Assume  $\gamma > 0$  is a small parameter. In this section, we reduce the Navier-Stokes dynamics to a system of ordinary differential equations following Section 4. Let  $E_N$  be the finite dimensional subspace  $E_N = \text{Span}\{e_0, e_1^+, \dots, e_N^+, e_1^-, \dots, e_N^-\}$  of the phase space  $\mathcal{H}$ , where  $e_j^\pm = (\omega_j^\pm, \theta_j^\pm)^{tr}$  are the eigenfunctions of the operator  $L$  with the zero eigenvalues. Let  $\mathbf{P}_N$  be a projection operator on  $E_N$  and  $\mathbf{Q}_N = \mathbf{I} - \mathbf{P}_N$ . The components of  $\mathbf{P}_N$  are defined by

$$(7.199) \quad \mathbf{P}_{1,N}v = \sum_{j=1}^N \langle \tilde{\omega}, \tilde{\omega}_j^+ \rangle \omega_j^+ + \sum_{j=1}^N \langle \tilde{\omega}, \tilde{\omega}_j^- \rangle \omega_j^-,$$

$$(7.200) \quad \mathbf{P}_{2,N}v = (2\pi h)^{-1} \langle w, 1 \rangle + \sum_{j=1}^N \langle w, \tilde{\theta}_j^+ \rangle \theta_j^+ + \sum_{j=1}^N \langle w, \tilde{\theta}_j^- \rangle \theta_j^-,$$

where  $v = (\tilde{\omega}, w)^{tr}$  and  $\tilde{e}_j^\pm = (\tilde{\omega}_j^\pm, \tilde{\theta}_j^\pm)$  are eigenfunctions of the conjugate operator  $L^*$  with zero eigenvalues  $\lambda = 0$  found in Sect. 6.4. Let us rewrite system (5.83), (5.84) as

$$(7.201) \quad \omega_t = \nu \Delta \omega - \{\psi, \omega\},$$

$$(7.202) \quad w_t = \Delta w - \{\psi, U + \gamma u_1 + w\} + \gamma^2 \eta_1.$$

First we transform equations (7.201)-(7.202) to a standard system with "fast" and "slow" modes. We follow Section 4. Let us introduce auxiliary functions  $R_\omega(X)$ ,  $R_\psi(X)$  and  $R_w(X)$  by

$$R_\omega(X) = \sum_{j=1}^N X_j^+ \omega_j^+ + \sum_{j=1}^N X_j^- \omega_j^-, \quad R_\psi(X) = \sum_{j=0}^N X_j^+ \psi_j^+ + \sum_{j=1}^N X_j^- \psi_j^-,$$

$$R_w(X) = X_0 + \sum_{j=1}^N X_j^+ \theta_j^+ + \sum_{j=1}^N X_j^- \theta_j^-,$$

and represent  $\omega, \psi$  and  $w$  by

$$(7.203) \quad \omega = \gamma R_\omega(X) + \hat{\omega}, \quad \psi = \gamma R_\psi(X) + \hat{\psi},$$

$$(7.204) \quad w = \gamma R_w(X) + \hat{w},$$

where  $\mathbf{P}_N(\hat{\omega}, \hat{w})^{tr} = 0$ ,  $X_i^\pm(t)$  are unknown functions,  $X = (X_0, X_1^+, \dots, X_N^+, X_1^-, \dots, X_N^-)^{tr}$ .

We substitute relations (7.203)-(7.204) in eqs. (7.201) and (7.202). After some transformations (following Section 4) one obtains the system

$$(7.205) \quad \frac{dX_i^\pm}{dt} = \gamma(G_i^\pm(X) + M_i^\pm(X) + f_i^\pm + F_i^\pm(X, \hat{\omega}, \hat{w}, \gamma)),$$

$$(7.206) \quad \hat{\omega}_t = \nu \Delta \hat{\omega} + \mathbf{P}_{1,N}F(X, \hat{\omega}, \hat{w}, \gamma),$$

$$(7.207) \quad \hat{w}_t = \Delta \hat{w} - \{\hat{\psi}, U\} + \mathbf{P}_{2,N}G(X, \hat{\omega}, \hat{w}, \gamma),$$

where in eqs. (7.206) and (7.207)

$$(7.208) \quad F = \{\gamma R_\psi(X) + \hat{\psi}, \gamma R_\omega(X) + \hat{\omega}\},$$

$$(7.209) \quad G = \{\gamma R_\psi(X) + \hat{\psi}, \gamma R_w(X) + \gamma u_1 + \hat{w}\} + \gamma^2 \eta_1.$$

The functions  $G_i^\pm(X)$  and  $M_i^\pm(X)$  give main contributions in the right hand sides of eqs. (7.205) and  $F_i^\pm$  are small corrections for small  $\gamma$ . One has

$$G_i^\pm(X) = \langle \{R_\psi(X), R_w(X)\}, \tilde{\theta}_i^\pm \rangle + \langle \{R_\psi(X), R_\omega(X)\}, \tilde{\omega}_i^\pm \rangle,$$

$$M_i^\pm(X) = \langle \{R_\psi(X), u_1\}, \tilde{\theta}_i^\pm \rangle.$$

These terms can be rewritten in a more explicit form as

$$(7.210) \quad G_i^+(X) = \sum_{j,l=1}^N G_{ijl}^{+++} X_j^+ X_l^+ + G_{ijl}^{--+} X_j^- X_l^-,$$

$$(7.211) \quad G_i^-(X) = \sum_{j,l=1}^N G_{ijl}^{+-+} X_j^+ X_l^-$$

and

$$(7.212) \quad M_i^+(X) = \sum_{j=1}^N M_{ij}^{++} X_j^+ + \sum_{j=1}^N M_{ij}^{+-} X_j^-,$$

$$(7.213) \quad M_i^-(X) = \sum_{j=1}^N M_{ij}^{-+} X_j^+ + \sum_{j=1}^N M_{ij}^{--} X_j^-.$$

Note that in eqs. (7.210) - (7.213) all the rest possible terms vanish since they are defined by integrals over  $x$  of functions odd in  $x$

The coefficients in (7.210), (7.211), (7.212) and (7.213) are defined by

$$(7.214) \quad M_{ij}^{\pm\pm}(u_1) = \langle \{\psi_j^\pm, \tilde{\theta}_i^\pm\}, u_1 \rangle,$$

$$(7.215) \quad G_{ijl}^{+++} = \langle \{\psi_j^+, \theta_l^+\}, \tilde{\theta}_i^+ \rangle + O(\nu^{-1})$$

$$(7.216) \quad G_{ijl}^{--+} = \langle \{\psi_j^-, \theta_l^-\}, \tilde{\theta}_i^+ \rangle + O(\nu^{-1})$$

$$(7.217) \quad G_{ijl}^{+-+} = \langle \{\psi_j^+, \theta_l^-\} + \{\psi_l^-, \theta_j^+\}, \tilde{\theta}_i^- \rangle + O(\nu^{-1})$$

for large  $\nu$ . Here we have used estimate (6.190), which implies that the terms, where  $\tilde{\omega}_j$  are involved, have the order  $O(\nu^{-1})$ . One has

$$(7.218) \quad f_i^\pm = \langle \eta_1, \tilde{\theta}_i^\pm \rangle.$$

The terms  $F_i^\pm$  are defined by

$$(7.219) \quad F_i^\pm = \gamma^{-1}(F_i^{\pm,\omega} + F_i^{\pm,w}),$$

where

$$F_i^{\pm,\omega} = \langle \{\gamma R_\psi(X), \hat{\omega}\} + \{\hat{\psi}, \gamma R_\omega(X) + \hat{\omega}\}, \tilde{\omega}_i^\pm \rangle,$$

$$F_i^{\pm,w} = \langle \{\gamma R_\psi(X), \hat{w}\} + \{\hat{\psi}, \gamma R_w(X) + \gamma u_1 + \hat{w}\}, \tilde{\theta}_i^\pm \rangle.$$

We consider equations (7.205), (7.206) and (7.207) in the domain (7.220)

$$\mathcal{D}_{\gamma, R_0, C_1, C_2, \alpha} = \{(X, \hat{w}, \hat{\omega}) : |X| < R_0, \|\hat{\omega}\|_\alpha < C_1 \gamma^{3/2}, \|\hat{w}\|_{1+\alpha} < C_2 \gamma^{3/2}\},$$

where  $\alpha > 3/4$ . Let us define the vector field  $V$  on the ball  $\mathcal{B}(R_0)^{2N} \subset \mathbf{R}^{2N}$  by

$$V(X) = (V_1^+(X), \dots, V_N^+(X), V_1^-(X), \dots, V_N^-(X))$$

LEMMA 7.1. *Let  $r \in (0, 1)$  and  $\alpha \in (3/4, 1)$ . Assume  $\gamma > 0$  is small enough:  $\gamma < \gamma_0(N, \nu, R_0, r, \alpha)$ . Then the local semiflow  $S^t$ , defined by equations (7.205), (7.206), and (7.207) has a locally invariant in the set  $\mathcal{D}_{\gamma, R_0, C_1, C_2, \alpha} \subset \mathcal{H}$  and locally attracting manifold  $\mathcal{M}_{2N+1, \gamma}$ . This manifold is defined by*

$$(7.221) \quad \hat{\omega} = \hat{\omega}_0(X, \gamma), \quad \hat{w} = \hat{w}_0(X, \gamma),$$

where  $\hat{\omega}_0(X, \gamma)$ ,  $\hat{w}_0(X, \gamma)$  are maps from the ball  $\mathcal{B}^{2N+1}(R_0) = \{X : |X| < R_0\}$  to  $H_\alpha$  and  $H_{1+\alpha}$  respectively, bounded in  $C^{1+r}$ -norm :

$$(7.222) \quad |\hat{\omega}_0(X, \gamma)|_{C^{1+r}(\mathcal{B}^{2N+1}(R_0))} < C_3 \gamma^2,$$

$$(7.223) \quad |\hat{w}_0(X, \gamma)|_{C^{1+r}(\mathcal{B}^{2N+1}(R_0))} < C_4 \gamma^2.$$

The restriction of the semiflow  $S^t$  on  $\mathcal{M}_{2N+1, \gamma}$  is defined by the vector field  $V(X)$ . The corresponding differential equations have the form

$$(7.224) \quad \frac{dX_i^\pm}{dt} = \gamma(V_i^\pm(X) + \phi_i^\pm(X, \gamma)),$$

where  $X = (X^+, X^-)$ ,

$$(7.225) \quad V_i^\pm(X) = G_i^\pm(X) + M_i^\pm(X) + f_i^\pm$$

$$(7.226) \quad \frac{dX_0}{dt} = 0,$$

and the corrections  $\phi_i^\pm(X, \gamma) = F_i^\pm(X, \hat{\omega}_0(X, \gamma), \hat{w}_0(X, \gamma), \gamma)$  satisfy the estimates

$$(7.227) \quad |\phi_i^\pm|, |D_X \phi_i^\pm| < c_1 \gamma^s, \quad s > 0.$$

This assertion is a consequence of Lemma 4.1. In coming sections we investigate system (7.224).

## 8. Quadratic systems

For sufficiently small  $\gamma$  we can remove small corrections  $\phi_i^\pm$  in the right hands of (7.224). Then we obtain a system of differential equations with quadratic nonlinearities. Let us consider a general class of such quadratic systems

$$(8.228) \quad \frac{dX}{dt} = K(X) + MX + g,$$

where  $X = (X_1, \dots, X_N)$ ,  $K = (K_1, \dots, K_N)$ ,  $g = (g_1, \dots, g_N) \in \mathbf{R}^N$ ,  $K(X)$  is a quadratic term defined by

$$K_i(X) = \sum_{j=1}^N \sum_{l=1}^N K_{ijl} X_j X_l,$$

and  $MX$  is a linear operator

$$(MX)_i = \sum_{j=1}^N M_{ij} X_j.$$

System (8.228) defines a local semiflow  $S^t(g, M)$  in the ball  $\mathcal{B}^N(R_0) \subset \mathbf{R}^N$  of the radius  $R_0$  centered at 0. We shall consider the vector  $g$  and the matrix  $M$  as parameters of this semiflow whereas the entries  $K_{ijl}$  will be fixed.

Let us formulate an assumption on entries  $K_{ijl}$ . We present  $X$  as a pair  $X = (Y, Z)$ , where

$$Y_l = X_l, \quad l \in I_p, \quad Z_j = X_{j+p}, \quad j \in J_p.$$

Here  $I_p = \{1, \dots, p\}$  and  $J_p = \{p+1, \dots, N\}$  are subsets of  $\{1, \dots, N\}$ . Then the system (8.228) can be rewritten as

$$(8.229) \quad \frac{dY}{dt} = K^{(1)}(Y) + K^{(2)}(Y, Z) + K^{(3)}(Z) + RY + PZ + f,$$

$$(8.230) \quad \frac{dZ}{dt} = \tilde{K}^{(1)}(Y) + \tilde{K}^{(2)}(Y, Z) + \tilde{K}^{(3)}(Z) + \tilde{R}Y + \tilde{P}Z + \tilde{f},$$

where for  $i = 1, \dots, p$

$$(8.231) \quad K_i^{(1)}(Y) = \sum_{j \in I_p} \sum_{l \in I_p} K_{ijl}^{(1)} Y_j Y_l, \quad K_i^{(3)}(Z) = \sum_{j \in J_p} \sum_{l \in J_p} K_{ijl}^{(3)} Z_j Z_l,$$

$$(8.232) \quad K_i^{(2)}(Y, Z) = \sum_{j \in I_p} \sum_{l \in J_p} K_{ijl}^{(2)} Y_j Z_l,$$

and for  $k = 1, \dots, N-p$

$$(8.233) \quad \tilde{K}_k^{(1)}(Y) = \sum_{j \in I_p} \sum_{l \in I_p} \tilde{K}_{kjl}^{(1)} Y_j Y_l, \quad \tilde{K}_k^{(3)}(Z) = \sum_{j \in J_p} \sum_{l \in J_p} \tilde{K}_{kjl}^{(3)} Z_j Z_l,$$

$$(8.234) \quad \tilde{K}_k^{(2)}(Y, Z) = \sum_{j \in I_p} \sum_{l \in J_p} \tilde{K}_{kjl}^{(2)} Y_j Z_l.$$

The linear terms  $MX$  take the form

$$(8.235) \quad (RY)_i = \sum_{j \in I_p} R_{ij} Y_j, \quad (\tilde{R}Y)_k = \sum_{j \in I_p} \tilde{R}_{kj} Y_j,$$

$$(8.236) \quad (PZ)_i = \sum_{j \in J_p} P_{ij} Z_j, \quad (\tilde{P}Z)_k = \sum_{j \in J_p} \tilde{P}_{kj} Z_j,$$

and  $f = (f_1, \dots, f_p)$ ,  $\tilde{f} = (\tilde{f}_1, \dots, \tilde{f}_{N-p})$ . We denote by  $S^t(\mathcal{P})$  the local semiflow defined by (8.229) and (8.230). Here  $\mathcal{P}$  is a semiflow parameter,  $\mathcal{P} = \{f, \tilde{f}, P, \tilde{P}, R, \tilde{R}\}$ . Let us formulate an assumption on quadratic terms  $K_i(X)$ .

**$p$ -Decomposition Condition** Suppose entries  $K_{ijl}$  satisfy the following condition. For some  $p$  there exists a decomposition  $Z = (X, Y)$ , where  $X \in \mathbf{R}^p$  and  $Y \in \mathbf{R}^{N-p}$  such that the linear system

$$(8.237) \quad \sum_{i \in J_p} \tilde{K}_{ijl}^{(1)} u_i = b_{jl}, \quad l, j \in I_p$$

has a solution  $u$  for all  $b_{jl}$ .

Clearly that for  $N > p^2 + p$  and generic matrices  $K$  this condition is valid.

Let us formulate some conditions to the matrices  $R, \tilde{R}, P$  and  $\tilde{P}$ . Let  $\epsilon > 0$  be a parameter. We suppose that

$$(8.238) \quad \tilde{P}_{ij} = -\xi^{-1} \delta_{ij}, \quad i = 1, \dots, N-p, \quad j = 1, \dots,$$

where  $\delta_{ij}$  is the Kronecker symbol,

$$(8.239) \quad \tilde{R}_{ij} = 0, \quad \tilde{f}_i = 0, \quad i = 1, \dots, N-p, \quad j = 1, \dots, p,$$

$$(8.240) \quad P_{ij} = \xi^{-1} T_{ij}, \quad |T_{ij}| < C_0, \quad i = 1, \dots, p, \quad j = 1, \dots, N - p,$$

$$(8.241) \quad |R_{ij}| < C, \quad i = 1, \dots, p, \quad j = 1, \dots, p,$$

LEMMA 8.1. Assume (8.238), (8.239), (8.240) and (8.241) hold and  $R_0 > 0$ . For sufficiently small positive  $\xi < \xi_0(R_0, r, f)$  the local semiflow  $S^t(\mathcal{P})$  defined by system (8.229), (8.230) has a locally invariant in the domain

$$(8.242) \quad \mathcal{D}_{R_0} = \{X : |Y| < R_0\}$$

and locally attracting manifold  $\mathcal{M}_{\mathcal{P}}$ . This manifold is defined by equations

$$(8.243) \quad Z = \xi(\tilde{K}^{(1)}(Y) + W(Y, \xi)), \quad Y \in \mathcal{B}^p(R_0)$$

where  $W$  is a  $C^1$  smooth map defined on the ball  $\mathcal{B}^p(R_0)$  to  $\mathbf{R}^{N-p}$  and such that

$$(8.244) \quad |W(\cdot, \xi)|_{C^1(\mathcal{B}^p(R_0))} < C_1 \xi^s, \quad s > 0.$$

**Proof.** Let us introduce a new variable  $w$  by

$$(8.245) \quad Z = \xi(\tilde{K}^{(1)}(Y) + w).$$

and the rescaled time by  $t = \xi\tau$ . Then for  $Y, w$  one obtains the following system

$$(8.246) \quad \frac{dY}{d\tau} = \xi G(Y, w, \xi),$$

$$(8.247) \quad \frac{dw}{d\tau} = \xi F(Y, w, \xi) - w,$$

where

$$(8.248) \quad \begin{aligned} G(Y, w, \xi) &= K^{(1)}(Y) + \xi K^{(2)}(Y, \tilde{K}^{(1)}(Y) + w) + \\ &+ \xi^2 K^{(3)}(\tilde{K}^{(1)}(Y) + w) + RY + T(\tilde{K}^{(1)}(Y) + w) + f, \\ F(Y, w, \xi) &= \tilde{K}^{(2)}(Y, \tilde{K}^{(1)}(Y) + w) + \\ &+ \xi \tilde{K}^{(3)}(\tilde{K}^{(1)}(Y) + w) + h(Y, w, \xi), \\ h(Y, w, \xi) &= -(D_Y \tilde{K}^{(1)}(Y))G(Y, w, \xi). \end{aligned}$$

Equations (8.246), (8.247) form a typical system involving slow ( $Y$ ) and fast ( $w$ ) variables. Existence of a locally invariant manifold for this system can be shown by the well known results (see [13, 3, 4, 1, 32, 38, 37]). The proof is standard, follows the scheme of Appendix 2 and we omit it.

The semiflow  $S^t$  restricted to  $\mathcal{M}$  is defined by the equations

$$(8.249) \quad \frac{dY}{d\tau} = \xi F(Y, \xi),$$

where

$$\begin{aligned} F(Y, \xi) &= K^{(1)}(Y) + \xi K^{(2)}(Y, \tilde{K}^{(1)}(Y) + W(Y, \xi)) + \\ &+ \xi^2 K^{(3)}(\tilde{K}^{(1)}(Y) + W(Y, \xi)) + RY + T\tilde{K}^{(1)}(Y) + W(Y, \xi) + f. \end{aligned}$$

The estimates for  $W$  show that  $F$  can be presented as

$$(8.250) \quad F(Y, \xi) = K^{(1)}(Y) + RY + T\tilde{K}^{(1)}(Y) + f + \phi_\xi(Y)$$

where a small correction  $\phi_\xi$  satisfies

$$(8.251) \quad |\phi_\xi|_{C^1(\mathcal{B}^p(R_0))} < c_0 \xi^{1/2}.$$

In (8.250)  $R$  and  $f$  are free parameters. The quadratic form  $D(Y) = K^{(1)} + T\tilde{K}^{(1)}$  can be also considered as a free parameter according to  $p$ -Decomposition Condition. Therefore, we have proved the following assertion.

LEMMA 8.2. *Let*

$$(8.252) \quad F(Y) = D(Y) + RY + f$$

*be a quadratic vector field on  $\mathcal{B}^p(R_0)$ , where*

$$D_i(Y) = \sum_{j=1}^p \sum_{l=1}^p D_{ijl} Y_j Y_l, \quad (RY)_i = \sum_{j=1}^p R_{ij} Y_j.$$

*Consider system (8.229), (8.230). Let  $p$ -Decomposition Condition hold. Then for any  $\epsilon > 0$  the field  $F$  can be  $\epsilon$ -realized by the semiflow  $S^t(\mathcal{P})$  defined by system (8.229), (8.230), where parameters  $\mathcal{P}$  are the matrices  $P, R, \tilde{P}, \tilde{R}$  and the vectors  $f, \tilde{f}$ .*

By Lemma 8.2 and results [36] we prove the following assertion on realization of all vector fields by quadratic systems.

PROPOSITION 8.3. *Consider the semiflows defined by systems (8.228), where the triple  $\{N, M, g\}$  serves as a parameter  $\mathcal{P}$ , for each  $N$  the coefficients  $K_{ijl}$  with  $i, j, l \in \{1, \dots, N\}$  satisfy  $p$ -decomposition condition for a  $p$  such that  $N/2 < p^2 + p \leq N$ .*

*Then these semiflows enjoy the following property. For each integer  $n$ , each  $\epsilon > 0$  and each vector field  $Q$  satisfying (3.23) and (3.24), there exists a value of the parameter  $\mathcal{P}$  such that the corresponding system (8.228) defines a semiflow  $S_{\mathcal{P}}^t$ , which  $\epsilon$ -realizes the vector field  $Q$  on  $n$ -dimensional positively invariant manifold  $\mathcal{M}_{n,Q}$ .*

Below we apply this result to the semiflows defined by problem (2.14)-(2.21).

### 9. Control of linear terms in system (7.224)

In this section we first show that the matrices  $\mathbf{M}^{\pm\pm}$  involved in system (7.224) are completely controllable by the function  $u_1(x, y)$  and, thus, the linear terms in the right hand side of this system satisfy the LOD condition from Sect. 4.

**9.1. Control of matrix  $\mathbf{M}$  by  $u_1$ .** To calculate the entries of  $M_{ij}$  we take into account that this matrix can be decomposed to 4 blocks  $M_{ij}^{\pm\pm}$ , where

$$(9.253) \quad M_{ij}^{\pm\pm}(u_1(\cdot, \cdot)) = \langle \{\psi_j^{\pm}, \tilde{\theta}_i^{\pm}\}, u_1 \rangle.$$

A straight forward calculation by (6.195), (6.193) and (6.196) gives

$$(9.254) \quad M_{ij}^{++}(u_1) = a_{ij}^{++} \int_0^{2\pi} \int_0^h [\tilde{\zeta}_{ij}(y) \cos((i+j)x) + \zeta_{ij} \cos((i-j)x)] u_1(x, y) dx dy,$$

$$(9.255) \quad M_{ij}^{--}(u_1) = a_{ij}^{--} \int_0^{2\pi} \int_0^h [\tilde{\zeta}_{ij}(y) \cos((i+j)x) - \zeta_{ij}(y) \cos((i-j)x)] u_1(x, y) dx dy,$$

and

$$(9.256) \quad M_{ij}^{+-}(u_1) = a_{ij}^{+-} \int_0^{2\pi} \int_0^h [\tilde{\zeta}_{ij}(y) \sin((i+j)x) + \zeta_{ij}(y) \sin((i-j)x)] u_1(x, y) dx dy,$$



(9.257)

$$M_{ij}^{-+}(u_1) = a_{ij}^{-+} \int_0^{2\pi} \int_0^h [-\tilde{\zeta}_{ij}(y) \sin((i+j)x) + \zeta_{ij}(y) \sin((i-j)x)] u_1(x, y) dx dy,$$

where

$$\zeta_{ij} = j \Psi_j(y) \frac{d\bar{\theta}_i(y)}{dy} + i \frac{d\Psi_j(y)}{dy} \bar{\theta}_i(y),$$

$$\tilde{\zeta}_{ij} = j \Psi_j(y) \frac{d\bar{\theta}_i(y)}{dy} - i \frac{d\Psi_j(y)}{dy} \bar{\theta}_i(y),$$

and  $a_{ij}^{\pm\pm}$  are coefficients such that  $|a_{ij}^{\pm\pm}| = 1/2$ . Using relations (6.188) and (6.176) for  $\bar{\theta}_i$  and  $\Psi_j$ , one obtains

$$(9.258) \quad \tilde{\zeta}_{ij} = \bar{a}_i \bar{b}_j \tilde{\eta}_{ij}, \quad \zeta_{ij} = \bar{a}_i \bar{b}_j \eta_{ij},$$

where  $\bar{a}_i, \bar{b}_j$  are some non-zero coefficients, and

(9.259)

$$\eta_{ij} = i j y (\sinh(i+j)(h-y)) + \frac{h i j \cosh(h i - (i+j)y)}{\sinh(j h)} - i \cosh(j(h-y)) \cosh(i(h-y)),$$

(9.260)

$$\tilde{\eta}_{ij} = i j y (\sinh(i-j)(h-y)) - \frac{h i j \cosh(h i - (i-j)y)}{\sinh(j h)} + i \cosh(j(h-y)) \cosh(i(h-y)).$$

LEMMA 9.1. *Given a quadruple of  $N \times N$  matrices  $T^{\pm\pm}$  with entries  $T_{ij}^{\pm\pm}$  there exists a  $2\pi$ -periodic in  $x$ , smooth function  $u_1(x, y)$  such that the support of  $u_1$  lies in the strip  $\delta_1 < y < h$  and*

$$(9.261) \quad M_{jl}^{++}[u_1] = T_{jl}^{++}, \quad M_{jl}^{--}[u_1] = T_{jl}^{--},$$

$$(9.262) \quad M_{jl}^{+-}[u_1] = T_{jl}^{+-}, \quad M_{jl}^{-+}[u_1] = T_{jl}^{-+},$$

where  $j, l = 1, 2, \dots, N$ .

**Proof.** Let us show that systems of equations (9.261) and (9.262) are resolvable. These two systems are independent, and we consider the first one, the second one can be treated in a similar way. We represent  $u_1(x, y)$  by a Fourier series:

$$u_1(x, y) = u_0^+(y) + \sum_{k=1}^{+\infty} \hat{u}_k^+(y) \cos(kx) + \hat{u}_k^-(y) \sin(kx).$$

By elementary transformations, we reduce (9.261) to the following form

$$(9.263) \quad \int_0^h \tilde{\eta}_{ij}(y) \hat{u}_{i+j}^+(y) dy = A_{ij}, \quad 1 \leq i, j \leq N,$$

$$(9.264) \quad \int_0^h \eta_{ij}(y) \hat{u}_{j-i}^+(y) dy = B_{ij}, \quad N \geq j \geq i \geq 1,$$

$$(9.265) \quad \int_0^h \eta_{ij}(y) \hat{u}_{i-j}^+(y) dy = C_{ij}, \quad N \geq i > j \geq 1,$$

where  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{C}$  are some matrices. In (9.263) we introduce an index  $m$  by  $m = i + j$ , in (9.264) by  $m = j - i$  and in (9.265) by  $m = i - j$ . These equations become

$$(9.266) \quad \int_0^h \tilde{\eta}_{i,m-i}(y) \hat{u}_m^+(y) dy = A_{i,m-i}, \quad 1 \leq i < m,$$

where  $m \in \{2, \dots, 2N\}$ ,

$$(9.267) \quad \int_0^h \eta_{i,i+m}(y) \hat{u}_m^+(y) dy = B_{i,m+i}, \quad 1 \leq i \leq N - m,$$

where  $m \in \{0, 1, \dots, N - 1\}$ ,

$$(9.268) \quad \int_0^h \eta_{i,i-m}(y) \hat{u}_m^+(y) dy = C_{i,i-m}, \quad m < i \leq N,$$

where  $m = 0, 1, \dots, N - 1$ .

Let us show that system (9.266)-(9.268) has a solution using the Fredholm alternative. The left hand sides of (9.266)-(9.268) define a linear map  $U$  from on the set of  $C^1$ -smooth functions  $\hat{u}_m(y)$  defined on the interval  $\delta_1 \leq y \leq h$  to a finite dimensional linear Euclidian space  $\mathcal{E}$ . Let us consider the image  $R(U)$  of  $U$ . If the closure  $Clos(R(U))$  does not coincide with the whole  $\mathcal{E}$ , then there is a nonzero vector from  $\mathcal{E}$  orthogonal to  $R(U)$ . Therefore, in this case there are coefficients  $X_{m,i}$ ,  $Y_{m,i}$  and  $Z_{m,i}$  such that

$$(9.269) \quad \sum_{i=1}^{m-1} |X_{m,i}| + \sum_{i=1}^{N-m} |Y_{m,i}| + \sum_{i=m+1}^N |Z_{m,i}| = 1,$$

$$(9.270) \quad S_m(y) + \tilde{S}_m(y) \equiv 0$$

for all  $y \in (\delta_1, h)$  and all  $m = \{1, \dots, N\}$ , where

$$(9.271) \quad \tilde{S}_m = \sum_{1 \leq i < m} X_{m,i} \tilde{\eta}_{i,m-i}(y),$$

$$(9.272) \quad S_m = \sum_{1 \leq i < N-m} Z_{m,i} \eta_{i,m+i}(y) + \sum_{m < i \leq N} Y_{m,i} \eta_{i,i-m}(y).$$

Since functions  $S_m, \tilde{S}_m$  are analytic in  $y$ , equation (9.270) is fulfilled for all  $y \in (-\infty, \infty)$ . Therefore, eq. (9.270) means that nontrivial linear combinations of the functions  $\eta_{i,m+i}(y)$ ,  $\tilde{\eta}_{i,m-i}(y)$  and  $\eta_{i,i-m}(y)$  with coefficients  $Z_{m,i}$ ,  $X_{m,i}$  and  $Y_{m,i}$  are zero for all  $y$  and  $m$ .

Let us prove that these nontrivial linear combinations do not exist. We see that

$$\eta_{i,m+i} = \frac{h(i+m)i}{\sinh((m+i)h)} \cosh(hi - (2i+m)y) + i(i+m)y \sinh(2i+m)(h-y) -$$

$$-i \cosh((m+i)(h-y)) \cosh i(h-y)$$

where  $m + 2i \in I_{m,N} = \{m + 2, m + 4, \dots, 2N - m\}$ ,

$$\eta_{i,i-m} = \frac{h(i-m)i}{\sinh((i-m)h)} \cosh(hi - (2i-m)y) + i(i-m)y \sinh(2i-m)(h-y) -$$

$$-i \cosh((i-m)(h-y)) \cosh i(h-y)$$

where  $2i - m \in \{m + 2, m + 4, \dots, 2N - m\}$ , and

$$\begin{aligned} \tilde{\eta}_{i,m-i} = & -\frac{h(m-i)i}{\sinh((m-i)h)} \cosh(hi + (m-2i)y) + i(m-i)y \sinh(2i-m)(h-y)) + \\ & + i \cosh((m-i)(h-y)) \cosh(i(h-y)), \end{aligned}$$

where  $m - 2i \in J_m = \{m - 2, m - 4, \dots, -m + 2\}$ . Let us observe that functions  $\cosh(a + k_1 y)$ ,  $\cosh(b + k_2 y)$ ,  $\sinh(a' + k_1 y)$ ,  $\sinh(b' + k_2 y)$  and  $y \sinh(c + k_1 y)$  are linearly independent for all  $a, b, a', b', c$ , if  $|k_1| \neq |k_2|$ . The sets  $I_{m,N}$  and  $J_m$  are disjoint. Thus the functions  $S_m$  and  $\tilde{S}_m$  are mutually linearly independent for any choice of coefficients  $X_{ij}, Y_{ij}$  and  $Z_{ij}$  such that  $\sum_{1 \leq i < m} X_{m,i} > 0$ , and  $\sum_{1 \leq i < N-m} |Z_{m,i}| + \sum_{m < i \leq N} |Y_{m,i}| > 0$ . We conclude thus that either

$$(9.273) \quad \tilde{S}_m(y) \equiv 0, \quad y \in (-\infty, \infty),$$

or

$$(9.274) \quad S_m(y) \equiv 0 \quad y \in (-\infty, \infty).$$

Let us consider the case when (9.273) holds. Consider the function  $\tilde{S}_m$  and terms proportional to  $y$  in this function. Suppose that there is a coefficient  $X_{m,i} \neq 0$ . We notice then, taking into account only the terms  $i(m-i)y \sinh(2i-m)(h-y)$  that (9.273) entails

$$X_{m,i} = X_{m,m-i}.$$

This relation implies, by the aforementioned linear independency of hyperbolic functions, that the identities

$$(9.275) \quad X_{m,i} w_{i,m}(y) = 0, \quad \text{for all } y \in \mathbf{R},$$

hold for each  $i \in \{\frac{m}{2} - 1, \dots, m - 1\}$  and  $m$ . Here

$$\begin{aligned} w_{i,m}(y) = & -\frac{h(m-i)i}{\sinh((m-i)h)} \cosh(hi + (m-2i)y) - \\ & -\frac{h(m-i)i}{\sinh(ih)} \cosh(hi + (i-2m)y) + m \cosh((m-i)(h-y)) \cosh(i(h-y)) \end{aligned}$$

Let us show that the function  $w_{i,m}(y) \neq 0$  for some  $y$ . This function can be represented as a sum of exponents  $\exp((m-2i)y)$ ,  $\exp(-(m-2i)y)$ ,  $\exp(my)$  and  $\exp(-my)$  with some coefficients. The coefficient at  $\exp(my)$  is not zero, therefore,  $w_{i,m}(y)$  is not zero for some  $y$ . Then eq.(9.275) implies that all coefficients  $X_{m,i} = 0$ .

Let us consider now the case when relation (9.274) holds. We consider in  $S_m(y)$  terms proportional to  $y$ . This gives  $Y_{m,i+m} = -Z_{m,i}$ . This relation entails

$$(9.276) \quad Z_{m,i} v_{i,m}(y) = 0, \quad \text{for all } y \in \mathbf{R},$$

that hold for all  $i = 1, 2, \dots, m$  and  $m$ . Here

$$\begin{aligned} v_{i,m}(y) = & h(m+i)i \left( \frac{1}{\sinh ih} - \frac{1}{\sinh(i+m)h} \right) \cosh(hi - (2i+m)y) - \\ & - m \cosh(2i+m)(h-y). \end{aligned}$$

It is clear that  $v_{i,m}(y)$  is not zero for some  $y$ . Then eq. (9.276) entails  $Z_{m,i} = 0$  for all  $i, m$ . The proof of the lemma is complete.

Let us formulate a lemma about control  $f$  by  $\eta_1$ .

LEMMA 9.2. *Given vectors  $f^+ = (f_1^+, \dots, f_N^+)$  and  $f^- = (f_1^-, \dots, f_N^-)$ , there exists a smooth  $2\pi$ -periodic in  $x$  function  $\eta_1(x, y)$  with the support in the domain  $\{(x, y) : x \in (-\infty, +\infty), \delta_0 < y < h\}$ , where  $\delta_0 \in (0, h/2)$ , such that*

$$\langle \tilde{\theta}_i^\pm, \eta_1 \rangle = f_i^\pm, \quad i = 1, \dots, N.$$

We omit an elementary proof.

**9.2. Verification of  $p$ -Decomposition condition for system (7.215), (7.216).** To complete the proof of main result, it is necessary to verify  $p$ -Decomposition condition from Section 4. We choose the set  $I_p$  and  $i_l$  with  $l = 1, \dots, p$  in  $p$ -Decomposition condition by  $i_l = k_l$ ,  $I_p = \{k_1, \dots, k_p\}$ , where  $k_i \in \{1, \dots, N\}$ . Let us set  $Y_l = X_{k_l}^+$ . Respectively, all the rest variables  $X_i^-$  and  $X_j^+$  with  $j \neq k_l$  will be  $Z_l$ . Let us verify relation (8.237). Note that the matrix  $\tilde{K}^{(1)}$  involves  $G_{jli}^{+++}$  and  $G_{jli}^{+-}$ . Therefore it is sufficient to verify that the linear system

$$(9.277) \quad \sum_{i \in J_p} G_{ijl}^{+++} u_i = b_{jl}, \quad j, l \in I_p$$

has a solution for any given  $b_{jl}$ . To check it, let us calculate the coefficients  $G_{ijl}^{+++}$  defined by (7.215). Integrating by parts one has

$$(9.278) \quad G_{ijl}^{+++} = -\langle \{\psi_j^+, \tilde{\theta}_i^+\}, \theta_l^+ \rangle + O(\nu^{-1}),$$

thus by definition (7.214) of  $M_{ij}^{\pm\pm}$  one has

$$(9.279) \quad G_{ijl}^{+++} = -M_{ij}^{++}(\theta_l^+) + O(\nu^{-1}).$$

Using this relation and (9.254) one obtains

$$2G_{ijl}^{+++} = \int_0^{2\pi} \int_0^h [\tilde{\zeta}_{ij}(y) \cos((j+i)x) + \zeta_{ij}(y) \cos((j-i)x)] \Theta_l(y) \cos(lx) dx dy + O(\nu^{-s}),$$

where  $s > 0$ . We can transform this relation to the form

$$(9.280) \quad G_{ijl}^{+++} = \frac{1}{4}(\delta_{l,i+j} \tilde{I}_{ijl} + \delta_{l,i-j} I_{ijl} + \delta_{l,j-i} I_{ijl}) + O(\nu^{-s_1}), \quad s_1 > 0,$$

where  $\delta_{i,j}$  denotes the Kronecker symbol and

$$\tilde{I}_{ijl} = \int_0^h \tilde{\zeta}_{ij}(y) \Theta_l(y) dy, \quad I_{ijl} = \int_0^h \zeta_{ij}(y) \Theta_l(y) dy, \quad 1 \leq i, j \leq N$$

Let us estimate  $\tilde{I}_{ij,i+j}$  and  $I_{ijl}$  for  $l = i - j$  and  $l = j - i$ . We compute these integrals taking into account  $\nu \gg 1, h = \log \nu$  in relations (9.258), (9.259) and (9.260) for  $\xi_{ij}, \tilde{\xi}_{ij}$ . As  $\nu \rightarrow +\infty$  we have the asymptotics

$$\xi_{ij} = a_i b_j (2ijy - i)(1 + O(\nu^{-s})),$$

$$\tilde{\xi}_{ij} = a_i b_j i(1 + O(\nu^{-s})),$$

where constants  $a_i, b_j$  do not depend on  $\nu$  and  $s > 0$ . For  $\Theta_j$  we use relation (6.177). We obtain then

$$(9.281) \quad \tilde{I}_{ij,i+j} = a_i b_j \left( \frac{i}{2(j+i)} + O(\nu^{-s}) \right), \quad \nu \rightarrow +\infty,$$

$$(9.282) \quad I_{ij,i-j} = a_i b_j \left( \frac{j-i}{2i} + O(\nu^{-s}) \right), \quad \nu \rightarrow +\infty, \quad i \geq j$$

and

$$(9.283) \quad I_{ij,j-i} = O(\nu^{-s}), \quad \nu \rightarrow +\infty, \quad j \geq i.$$

Let us apply now Lemma 8.2. For each integer  $p$  and a large  $N = N(p)$  we solve system (9.277). By (9.280) this system can be reduced to the following form:

$$(9.284) \quad u_{k_i+k_j} \tilde{I}_{k_i,k_j,k_i+k_j} + u_{k_i-k_j} I_{k_i,k_j,k_i-k_j} = b_{ij}, \quad i, j = 1, \dots, p, k_i > k_j$$

$$(9.285) \quad u_{k_i+k_j} \tilde{I}_{k_i,k_j,k_i+k_j} + u_{k_j-k_i} I_{k_i,k_j,k_j-k_i} = b_{ij}, \quad i, j = 1, \dots, p, k_j > k_i$$

$$(9.286) \quad u_{2k_i} \tilde{I}_{k_i,k_i,2k_i} = b_{ii}, \quad i = j = 1, \dots, p.$$

We use a lemma.

LEMMA 9.3. *There exists a subset  $\mathcal{K}_p$  of  $\{1, \dots, N\}$  satisfying the following conditions:*

(Ki)

$$k_1 < k_2 < \dots < k_p, \quad k_i \equiv 1 \pmod{3},$$

(Kii) *all the sums  $k_i + k_j$  are mutually distinct, i.e.,  $k_i + k_j = k_{i'} + k_{j'}$  implies  $i = i', j = j'$  (in other words, the map  $(i, j) \rightarrow k_i + k_j$  from  $\{1, \dots, p\} \times \{1, \dots, p\}$  to  $\{1, \dots, 2N\}$  is injective).*

**Proof.** The set  $\mathcal{K}_p = \{k_1, k_2, \dots, k_p\}$  of indices satisfying these conditions can be found by an iterative procedure. Let us set  $k_1 = 1, k_2 = 7$ . We take a sufficiently large  $k_3 = 3m_3 + 1$  such that  $3k_2 < k_3$  (for example,  $k_3 = 19$ ). At  $j + 1$ -th step we take an odd  $k_j$  such that  $k_{j+1} > 3k_j$ . Notice that at  $j$ -th step all possible sums form a subset of the set  $\{2, \dots, 2k_j\}$  and all possible differences give a subset of the set  $\{1, \dots, k_j\}$ . Therefore, the new sums  $k_{j+1} + k_l$  do not coincide with the sums obtained earlier. This shows that the inductive process can be continued, and completes the proof.

By this lemma we can prove resolvability of system (9.284)-(9.286). Using **Ki** we observe that  $k_i + k_j \neq k_l - k_s$  for all  $i, j, l, s \in \{1, \dots, p\}$ . Due to this fact, we can put  $u_{k_i-k_j} = 0$  for all  $i, j$  that simplifies system (9.284)-(9.286) and gives

$$(9.287) \quad u_{k_i+k_j} \tilde{I}_{k_i,k_j,k_i+k_j} = b_{ij}, \quad i, j = 1, \dots, p.$$

By (9.281),(9.282) the coefficients  $\tilde{I}_{k_i,k_j,k_i+k_j}$  are not zero for  $i \neq j$ . Then due to **Kii** equation (9.287) has a solution and thus  $p$ -Decomposition condition is fulfilled.

## 10. Proof of main results

Let us prove Theorem 3.2. To make our arguments more transparent, we describe here a sequence of steps for  $\epsilon$ -realization of a vector field  $Q$  (see (3.22)) on the unit ball  $\mathcal{B}^n$ . We suppose  $Q$  satisfy (3.23) and (3.24). Our goal is to find parameters  $\mathcal{P}$  of IBVP (2.14) -(2.21) such that the corresponding family of semiflows, generated by this IBVP,  $\epsilon$ -realizes  $Q$ . We proceed it in two steps.

*Step 1.* Using proposition (8.3), for  $\epsilon_0 > 0$  we  $\epsilon_0$ -realize the field  $Q$  by the family of the semiflows defined by quadratic vector fields (8.228) on  $\mathbf{R}^N$  satisfying  $p$ -decomposition condition and with parameters  $N, M, g$ .

Lemma 9.3 shows that systems (7.224) satisfy  $p$ -decomposition condition, thus, the corresponding semiflows  $\epsilon$ -realize  $Q$ . The corresponding positively invariant manifold  $\mathcal{M}_{n,Q}$  is diffeomorphic to the unit ball  $\mathcal{B}^n$ . The right hand sides of system

(7.224) is defined by a quadratic field  $V$ , which can be represented as  $V(X) = K(X) + MX + g$ , (see (8.228) with some  $K = K(Q, \epsilon_0)$ ,  $M = M(Q, \epsilon_0)$  and  $g = g(Q, \epsilon_0)$ ).

*Step 2.* Consider the field  $V$  defined via  $N$ ,  $K$ ,  $M$  and  $g$  and obtained at the previous step. For each  $V$  and  $\epsilon_1 > 0$  we find a IBVP (2.14) -(2.21) such that the corresponding semiflow defined by this problem  $\epsilon_1$ -realizes the vector field  $V$  on the locally invariant in the domain  $\mathcal{D}_{\gamma, R_0, C_1, C_2, \alpha} \subset \mathcal{H}$  and locally attracting manifold  $\mathcal{M}_{2N+1, \gamma}$  of dimension  $2N + 1$ . To proceed it, we use the following parameters: functions  $U(y)$ ,  $u_1(x, y)$  satisfying (5.80), (5.81),  $\eta(x, y)$  and the numbers  $\nu, \gamma$ . Here we use the method of Section 4. We first choose a sufficiently large  $\nu > \nu_0$  (in order to use Proposition 6.2 about the spectrum of the linear operator  $L$ ). Then for sufficiently small positive  $\gamma$  such that  $\gamma < \gamma_0(R_0, \nu, \epsilon_1, \alpha, N, K, M, g)$  we can reduce the semiflow defined by the IBVP (2.14) -(2.21) to system (7.224). For sufficient small  $\gamma$  we can remove corrections  $\phi_i$  in the right hand sides of (7.224). Note that by variations of  $u_1$  and  $\eta$  we can obtain all quadratic fields  $V = KX + MX + g$  with any prescribed  $M$  and  $g$  that follows from results of Sect. 9.1 (see Lemmas 9.1 and 9.2).

If  $R_0$  is large enough, we obtain the embeddings

$$\mathcal{M}_{n, Q} \subset \mathcal{M}_{2N+1, \gamma} \subset \mathcal{H},$$

where  $\mathcal{H}$  is the phase space defined by (5.54). For sufficiently small  $\epsilon_0, \epsilon_1$  steps 1 and 2 give  $\epsilon$ -realization of the field  $Q$  by semiflows defined by IBVP (2.14) -(2.21) with parameters  $U(\cdot), u_1(\cdot, \cdot), \eta(\cdot, \cdot), \nu, \gamma$ . Theorem 3.2 is proved.

**Proof of Corollary.** Consider a flow on finite dimensional smooth compact manifold defined by a  $C^1$ -smooth vector field and having a hyperbolic compact invariant set  $\Gamma$ . For some integer  $n > 0$  we can find a smooth vector field  $Q$  on a unit ball  $\mathcal{B}^n$ , which generates a semiflow having a topologically equivalent hyperbolic compact invariant set  $\Gamma'$  (and the corresponding restricted dynamics are orbitally topologically equivalent). Due to the Theorem on Persistence of Hyperbolic sets (see [28, 15]) we find a sufficiently small  $\epsilon(\Gamma', Q) > 0$  such that for all  $C^1$  perturbations  $\tilde{Q}$  of  $Q$  satisfying  $|\tilde{Q}|_{C^1(\mathcal{B}^n)} < \epsilon$  the perturbed vector fields  $Q + \tilde{Q}$  generate hyperbolic compact invariant sets  $\tilde{\Gamma}$  topologically equivalent to  $\Gamma$  (and the corresponding restricted dynamics are orbitally topologically equivalent). Then we  $\epsilon$ -realize this field by Theorem 3.2.

## 11. Conclusion

In the case of a free liquid surface in contact with air, buoyancy and surface tension effects interplay in the convection. In this paper we have considered the model, where buoyancy is zero, then we are dealing with the Marangoni effect and Marangoni -Bénard convection studied in a number of physical and mathematical works. It is shown that the corresponding Navier -Stokes equations can generate semiflows with complicated hyperbolic dynamics. They can realize, with arbitrary accuracy, any finite dimensional vector fields. This realization can be done on stable invariant manifolds (in general, these manifolds are not globally attracting). The main instrument in this realization is a choice of an external heat source and the Prandtl number. The author thinks that the methods of this paper work for the Boussinesq model with no-slip or free surface boundary conditions, but it is not clear how to apply the method of this paper for incompressible viscous fluids. Note

that, in Sections 4 and 8, a general method to prove existence of chaotic behaviour for quadratic systems of ODE's and PDE's is stated.

## 12. Acknowledgements

I dedicate this paper to memory of my friends Vladimir Shelkovich and Sergey Aristov. I am grateful to Prof. D. Grigoriev (Lille) for the help and discussions and A. Weber (Bonn) for interesting discussions. Also the author is thankful to Prof. V. Kozlov (Linköping) for critics (severe, but useful), and Prof. V. Volpert (Lyon) for his hospitality in Lyon University during 1998-2003 and fruitful discussions. Also I am grateful the anonymous Referees for useful remarks.

I am grateful to Prof. D. Ruelle, who has encouraged, in 1991, the first steps of the author in this domain.

This work was financially supported by Government of Russian Federation, Grant 074-U01.

## 13. Appendix 1

Let us consider the Fourier series for  $\bar{\omega}$  and  $\bar{\psi}$ . Let us represent  $u$  and  $Tr(u)$  by

$$(13.288) \quad u = \sum_{k \in \mathbf{Z}} \sum_{m=0}^{\infty} \hat{u}_{k,m} \cos(mh^{-1}y) \exp(ikx),$$

$$(13.289) \quad Tr(u) = \sum_{m=0}^{\infty} \hat{u}_k \exp(ikx), \quad \hat{u}_k = \sum_{m=0}^{\infty} \hat{u}_{k,m}.$$

Then the vorticity  $\bar{\omega}$ , defined by (5.55), (5.56), is

$$(13.290) \quad \bar{\omega} = \sum_{k \in \mathbf{Z}} ik \hat{u}_k \exp(ikx) \frac{\sinh(k(h-y))}{\sinh kh}.$$

We observe that

$$(13.291) \quad \|\bar{\omega}(u)\|^2 \leq C_1 \sum_{k \in \mathbf{Z}} |k| |\hat{u}_k|^2,$$

for some  $C_1 > 0$ . By (13.290) one obtains that the stream function  $\bar{\psi}$  has the form

$$(13.292) \quad \bar{\psi} = \sum_{k \in \mathbf{Z}} i \hat{u}_k \exp(ikx) \left( -\frac{y \cosh(k(h-y))}{2 \sinh kh} + \frac{h \sinh ky}{2(\sinh kh)^2} \right).$$

This relation implies

$$\|\nabla \bar{\psi}\|^2 \leq c_1 \sum_{k \in \mathbf{Z}} |k| |\hat{u}_k|^2 \leq c_2 \|u\|_{\alpha}, \quad \alpha > 1,$$

(the second inequality follows from embedding (5.53) for traces). Since

$$\sup_{y \in (0,h)} |ky| \cosh(k(h-y)) (\sinh kh)^{-1} \leq C_3,$$

one has

$$\|\nabla \bar{\psi}\| \leq C_1 \sum_{k \in \mathbf{Z}, k \neq 0} |\hat{u}_k| \leq \left( \sum_{k \in \mathbf{Z}, k \neq 0} |\hat{u}_k|^2 k^{2\gamma} \right)^{1/2} \left( \sum_{k \in \mathbf{Z}, k \neq 0} k^{-2\gamma} \right)^{1/2}$$

with  $\gamma > 1/2$ . The last estimate gives

$$|\nabla \bar{\psi}| \leq C_2 \|u\|_\alpha, \quad \alpha > 3/2.$$

## 14. Appendix 2

**Proof of Lemma 4.1.** This assertion is a consequence of Theorem 6.1.7 [13]. In the variables  $\tilde{w}$ ,  $X$  system (4.35), (4.36) takes the form

$$(14.293) \quad X_t = g(X, \tilde{w}),$$

where

$$(14.294) \quad \begin{aligned} g &= \gamma^{-1} \mathbf{P}_1 F(\gamma(X + w_0) + \tilde{w}) + \gamma \bar{f}_1, \\ \tilde{w}_t &= L\tilde{w} + f_0(X, \tilde{w}), \quad f_0 = \mathbf{P}_2 F(\gamma(X + w_0) + \tilde{w}). \end{aligned}$$

Using a standard truncation trick we first modify eq. (14.293) as follows:

$$(14.295) \quad X_t = G(x, \tilde{w}),$$

where

$$G(x, \tilde{w}) = g(X, \tilde{w}) \chi_{R_1}(X)$$

and  $\chi_{R_1}$  is a smooth function such that  $\chi_{R_1}(X) = 1$  for  $|X| < R_1$  and  $\chi_{R_1}(X) = 0$  for  $|X| > 2R_1$ . After this modification,  $X$ -trajectories of (14.295) are defined for all  $t \in (-\infty, +\infty)$  (as in Theorem 6.1.7 [13]). Then an invariant manifold for the semiflow defined by system (14.295), (14.294) is a locally invariant one for the semiflow generated by (14.293), (14.294).

Let us consider the semigroup  $\exp(Lt)$ . If  $w(0) \in B_2$  we have the following estimates [13]

$$\begin{aligned} \|\exp(Lt)w(0)\| &\leq C_0 \|w(0)\| \exp(-\beta t), \\ \|\exp(Lt)w(0)\|_\alpha &\leq C_1 \|w(0)\| t^{-\alpha} \exp(-\beta t), \end{aligned}$$

where  $M, \beta > 0$  do not depend on  $\gamma$ . Moreover,

$$(14.296) \quad M_0 = \sup_{(X, \tilde{w}) \in \mathcal{B}_{R, \gamma, C}} \|f\| < c_2 \gamma^2,$$

$$(14.297) \quad \lambda = \sup_{(X, \tilde{w}) \in \mathcal{B}_{R, \gamma, C}} \|D_X f_0\| + \|D_{\tilde{w}} f_0\| < c_3 \gamma^2,$$

$$(14.298) \quad M_2 = \sup_{(X, \tilde{w}) \in \mathcal{B}_{R, \gamma, C}} \|D_{\tilde{w}} g\| < c_4 \gamma,$$

We set  $\mu = \beta/4$ . Then for small  $\gamma$

$$(14.299) \quad M_3 = \sup_{(X, \tilde{w}) \in \mathcal{B}_{R, \gamma, C}} \|D_X g\| < c_4 \gamma < \mu.$$

We set  $\Delta = 2\theta_1$ , where

$$(14.300) \quad \theta_p = \lambda M_0 \int_0^\infty du u^{-\alpha} \exp(-(\beta - p\mu')u), \quad 1 \leq p \leq 1+r,$$

and  $\mu' = \mu + \Delta M_2$ . For sufficiently small  $\gamma$  one has  $\mu' < \beta/2$ , therefore, the integral in the right hand side of (14.300) converges and, according to (14.298), one obtains  $\theta < c_5 \gamma^2$  (since  $M$  is independent of  $\gamma$ ). We notice then that for sufficiently small  $\gamma$  the following estimates

$$\begin{aligned} (1+r)\mu' &< \beta/2, \\ \theta_1 &< \Delta(1+\Delta)^{-1}, \quad \theta_1 < 1, \quad \theta_1(1+\Delta)M_2\mu'^{-1} < 1, \end{aligned}$$



and

$$\theta_p(1 + \frac{(1 + \Delta)M_2}{r\mu'}) < 1$$

hold.

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